Matroids: algebraicity, duality, and valuations

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Algebraic matroids and Frobenius flocks, Bollen-D-Pendavingh
Matroids over one-dimensional groups, Bollen-Cartwright-D

From matrix to matroid I

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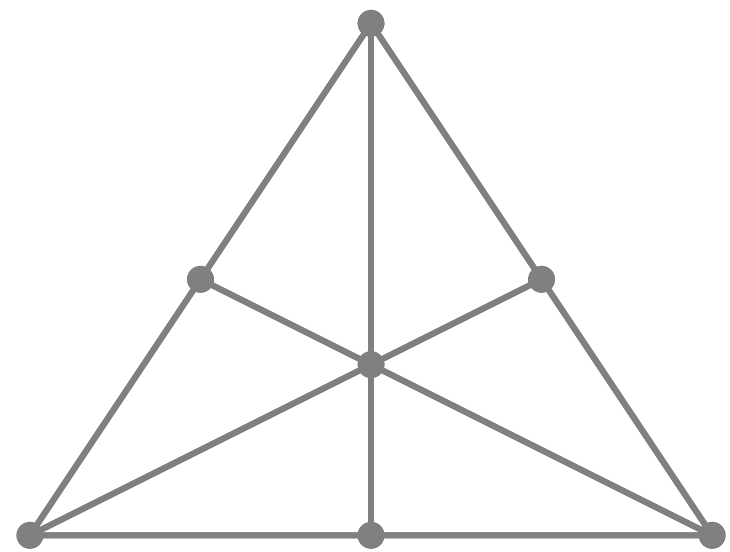

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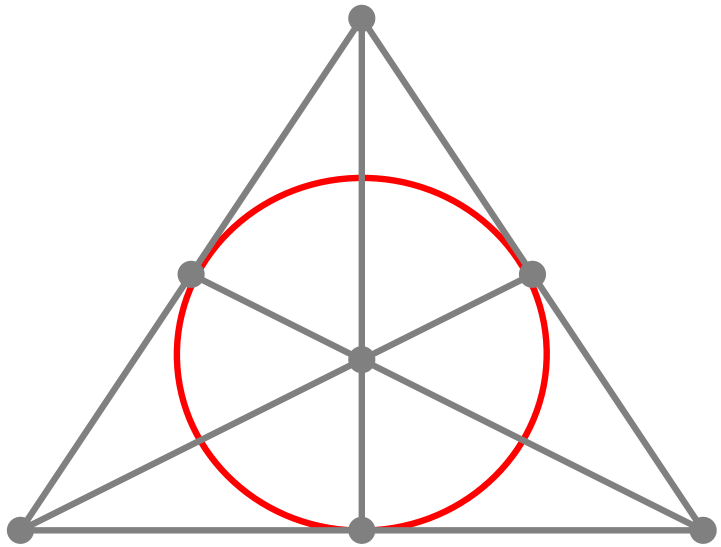

(non-)Fano matroid

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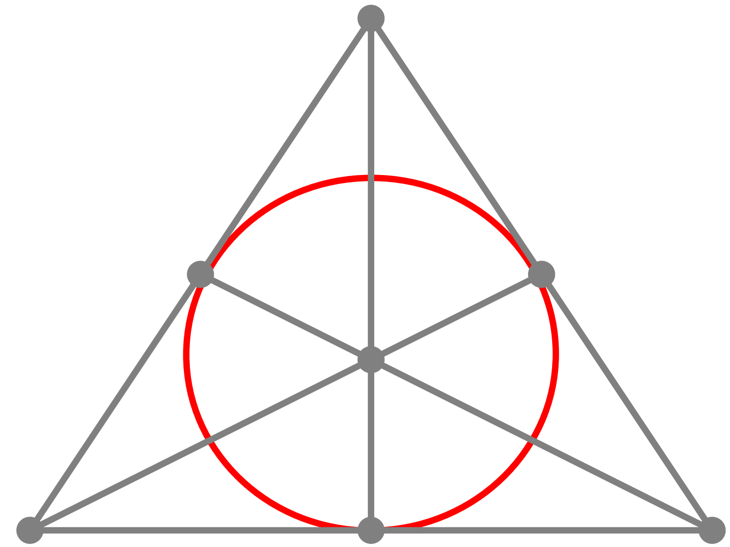

(non-)Fano matroid

This collection $\mathcal{I} \subseteq 2^{[n]}$ is nonempty, downward closed, and satisfies $\forall I, J \in \mathcal{I} : |J| > |I| \Rightarrow \exists j \in J \setminus I : I + j \in \mathcal{I}$; these are the defining properties of a *matroid* on $[n]$.

Well-understood breeds of matroids

Definition: A *matroid* on $[n]$ is a nonempty, downward closed collection $\mathcal{I} \subseteq 2^{[n]}$ s.t. $\forall I, J \in \mathcal{I} : |J| > |I| \Rightarrow \exists j \in J \setminus I : I + j \in \mathcal{I}$. The maximal *independent sets* are called the *bases*.

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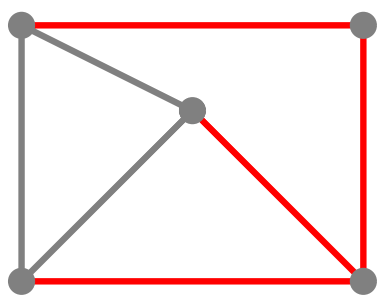
a basis: 

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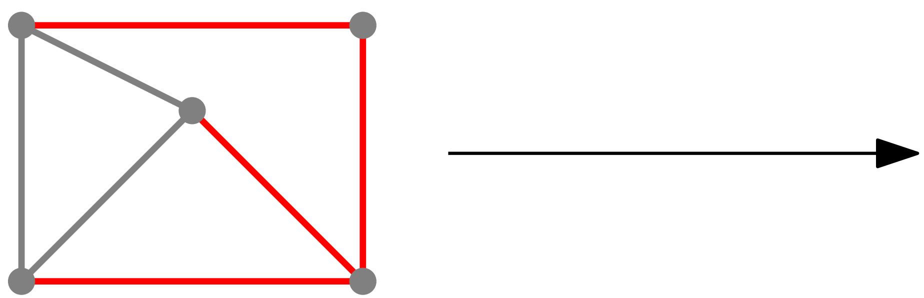

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Every graphical matroid is linear (over every field).

The ugly ducks among matroids

Definition: Let $L \supseteq K$ be a field extension and $x_1, \ldots, x_n \in L$. Set $\mathcal{I} := \{I \subseteq [n] : (x_i)_{i \in I}$ *algebraically independent* over $K\}$. Such a matroid is called *algebraic* (over K).

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{aligned} x_1 &= t_1 \\ x_2 &= t_2 \\ x_3 &= t_3 \\ x_4 &= t_2 + t_3 \\ x_5 &= t_1 + t_3 \\ x_6 &= t_1 + t_2 \\ x_7 &= t_1 + t_2 + t_3 \end{aligned}$$

$$L = K(t_1, t_2, t_3)$$

Why study algebraic matroids?

Generic completion

K algebraically closed

$X \subseteq K^n$ irreducible closed subvariety

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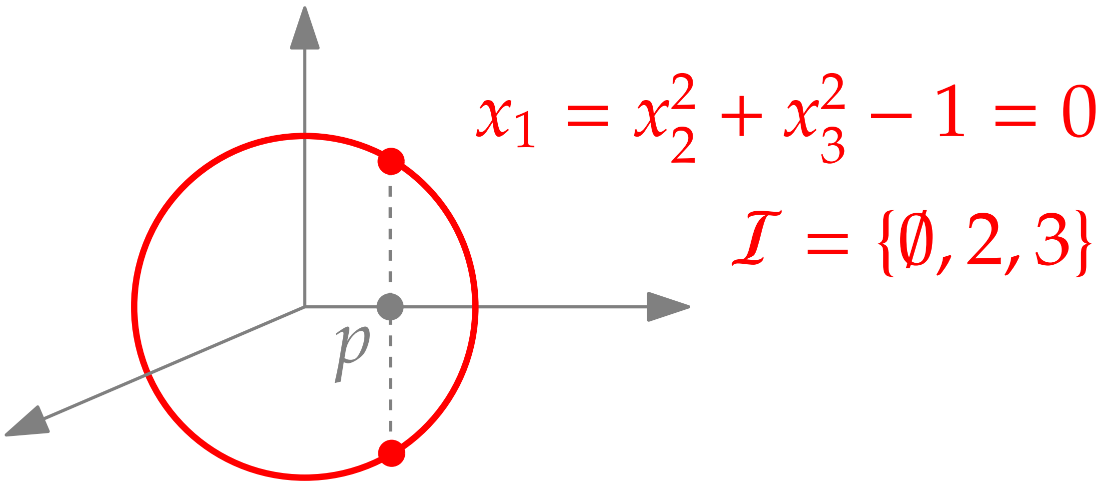

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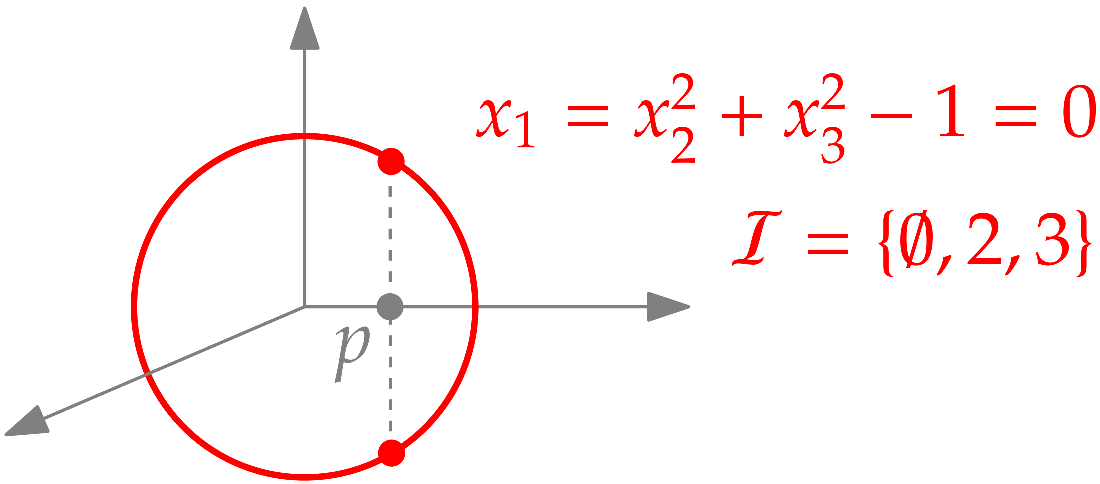

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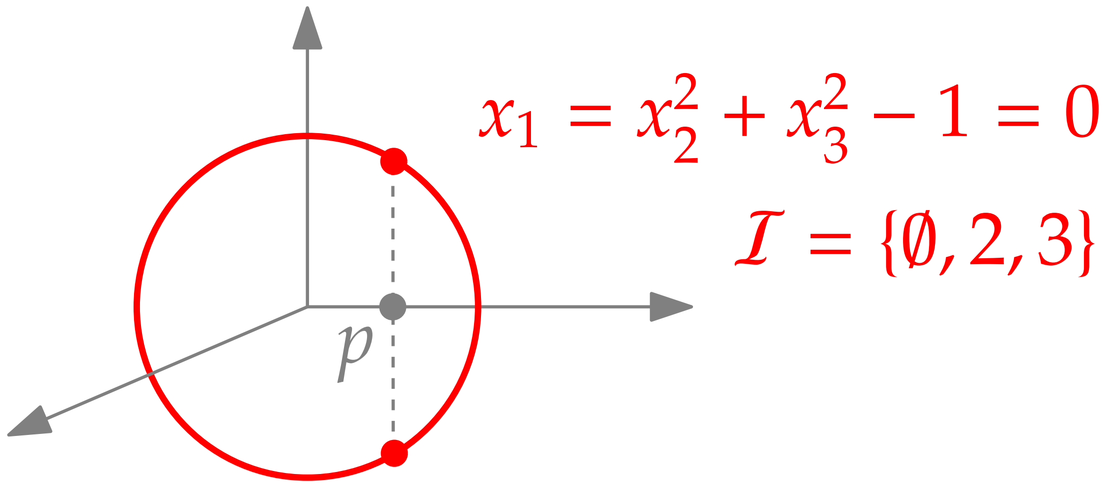

Problem: Given X and $I \subseteq [n]$, decide whether $I \in \mathcal{I}$.

Can be solved by Buchberger's algorithm for *elimination*, but this is not efficient.

Case study: generic low-rank matrix completion

$$[n] = [\ell] \times [m], \quad \ell, m \geq k, \quad K^{\ell \times m} \supseteq X := \{A \mid \mathrm{rk}\,(A) \leq k\}$$

Generic rank-*k* completion problem: On input $I \subseteq [\ell] \times [m]$, decide wether a generic choice of $(a_{ij})_{(i,j) \in I}$ can be completed to a matrix of rank $\leq k$.

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Open problem: is there a poly time deterministic algorithm that on input $S \subseteq \mathbb{Q}^n$ decides if S can be partitioned by a hyperplane into two linearly independent sets?

Linearising algebraic matroids

$X \subseteq K^n$ irreducible, and $q \in X$ smooth $\rightsquigarrow$ the *tangent space* $T_q X$ defines a matroid on $[n]$ with $\mathcal{I}(T_q X) \subseteq \mathcal{I}(X)$.

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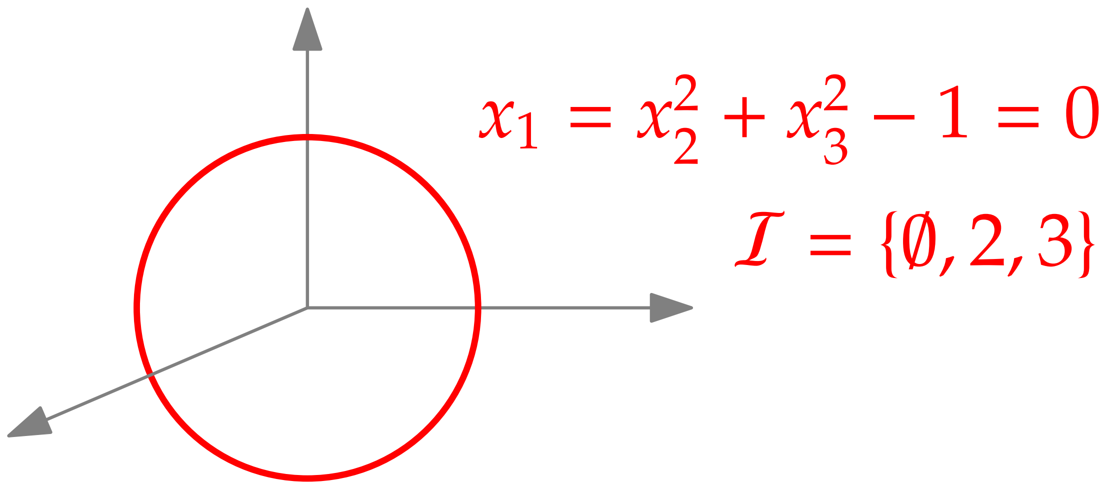

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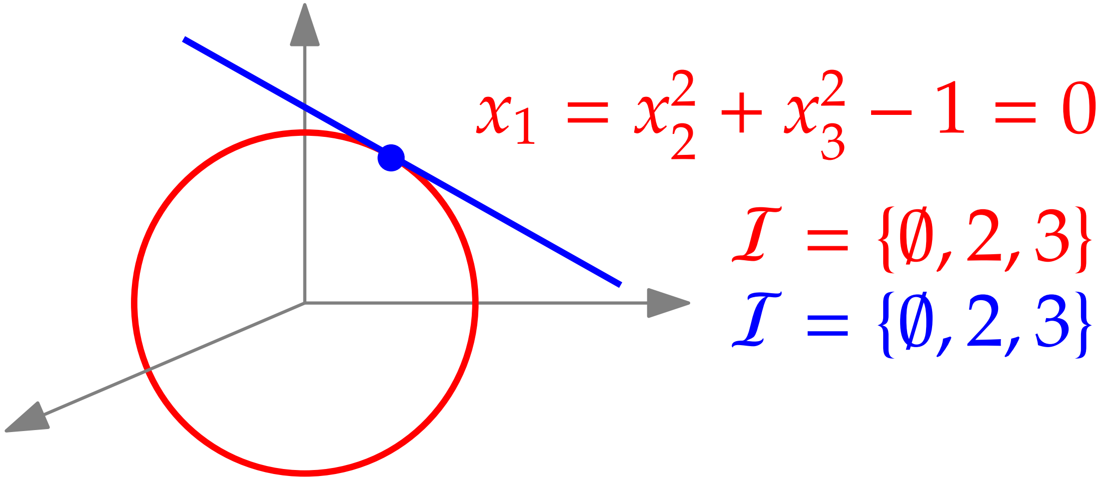

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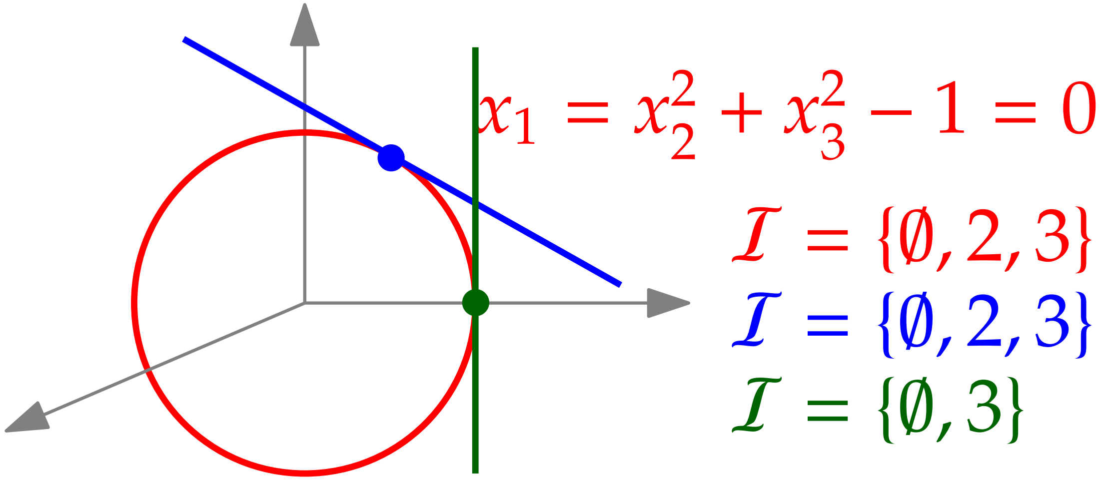

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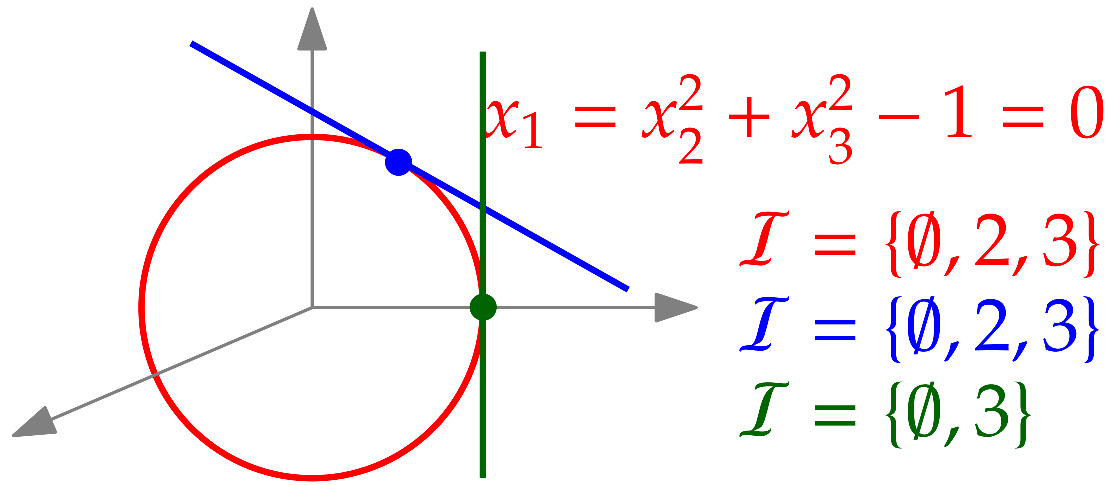

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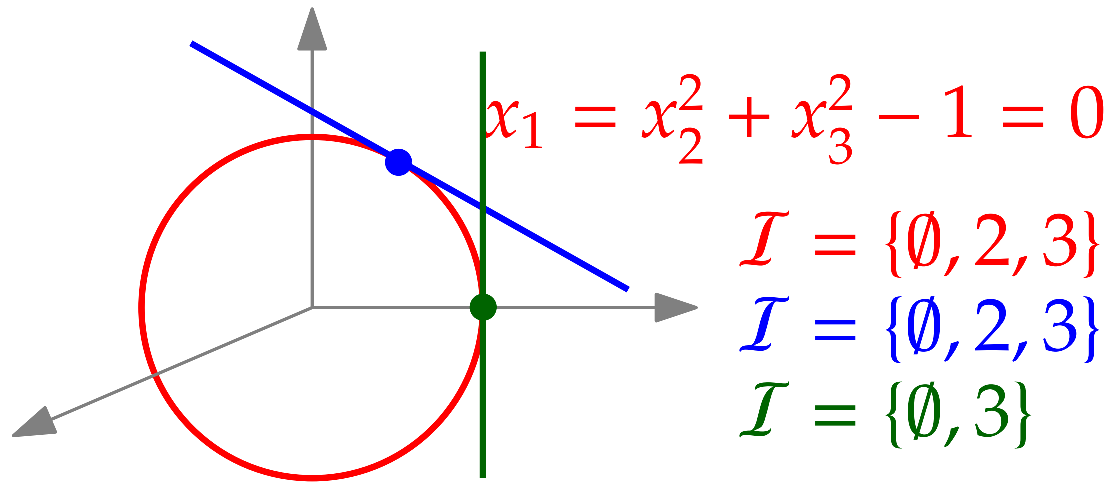

Consequences

- Algebraic matroids in characteristic 0 are linear (**Ingleton**)
- Sometimes there is an efficient probabilistic algorithm for the generic completion problem: sample $q \in X$, compute T_qX, and use Gaussian elimination to check $I \in \mathcal{I}(T_qX)$.

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$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{char} \neq 2} A^\perp = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

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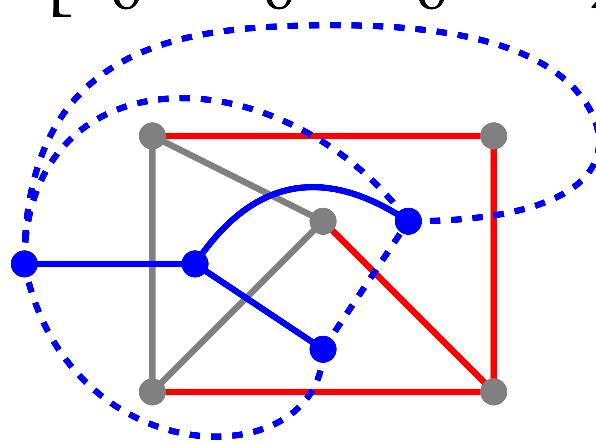
The dual of a *planar graph* matroid is graphical:



Main challenges on algebraic matroids

Testing algebraicity: given a matroid, how does one decide if it is algebraic?

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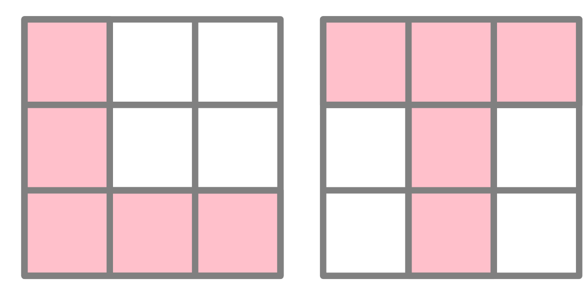
Example (Alfter-Hochstättler):
the *tic-tac-toe* matroid on $[3] \times [3]$ has as bases all quintuples *except* all 4 L's and all 4 T's. Is it algebraic?? Its dual is *not*.



From matrix to matroid (valuation) II

K a field, $v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ a *non-Archimedean valuation*:
$v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) + v(b)$, and $v(a + b) \geq \min\{v(a), v(b)\}$

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 8 \end{bmatrix} \xrightarrow{K = \mathbb{Q},\, v = \text{2-adic}} \begin{array}{l} \mu(\{1,2\}) = \mu(\{1,3\} \\ = \mu(\{2,3\}) = \mu(\{2,4\}) \\ = \mu(\{3,4\}) = 0 \\ \mu(\{1,4\}) = 3 \end{array}$$

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This *matroid valuation* $\mu : \binom{[n]}{d} \to \overline{\mathbb{R}}$ satisfies: $\mu \neq \infty$ and

$\forall B, B', i \in B \backslash B' \exists j \in B' \backslash B : \mu(B) + \mu(B') \geq \mu(B - i + j) + \mu(B' + i - j)$.

Matroid valuations play the role of linear spaces in trop geometry.

The Lindström valuation

Definition (Bollen-D-Pendavingh, Cartwright)
K an algebraically closed field of characteristic $p > 0$
$L = K(x_1, \dots, x_n) \supseteq K$ of transcendence degree d
$\rightsquigarrow \mu : \binom{[n]}{d} \to \overline{\mathbb{R}}$ defined as $\mu(I) := \log_p[L : K((x_i)_{i\in I})^{\text{sep}}]$
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Theorem (B-D-P): if the Lindström valuation is trivial, i.e. $\exists \alpha \in \mathbb{Z}^n$: for all bases $\mu(B) = \sum_{i \in B} \alpha_i$, then the algebraic matroid is also linear.

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Corollary: Matroids, such as Fano, that admit only trivial valuations are algebraic over K iff they are linear over K.

Bollen used enhancements of this for ruling out algebraicity of many matroids on ≤ 9 elements.

Matroids over one-dimensional groups

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Key to the solution: the *endomorphism ring* $\mathbb{E}$ *of* G:
- $K[F]$ with $Fa = a^p F$ if $G = (K, +)$;
- $\mathbb{Z}$ if $G = (K^*, \cdot)$; and
- $\mathbb{Z}$ or an order in an imaginary quadratic number field or in a quaternion algebra if $G =$ an elliptic curve.

In all cases, $\mathbb{E}$ is an Ore ring, hence generates a skew field Q.

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The closed subgroup $X \subseteq G^n$ is the image of a map $G^d \to G^n$ given by a rank-d matrix $A \in \mathbb{E}^{n \times d}$, and is uniquely determined by the column span of A, a right Q-subspace of Q^n.

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Theorem (B-Cartwright-D)
The Lindström valuation of the matroid defined by X maps $I \subseteq [n]$ of size d to v(*Diedonné determinant* of $A[I]$).

From matrix to matroid (valuation) III

The closed subgroup $X \subseteq G^n$ is the image of a map $G^d \to G^n$ given by a rank-d matrix $A \in \mathbb{E}^{n\times d}$, and is uniquely determined by the column span of A, a right Q-subspace of Q^n.

The ring $\mathbb{E}$ comes with a valuation: $v(\alpha)$ is the degree of inseparability of $\alpha : G \to G$; this extends to $v : Q \to \overline{\mathbb{R}}$.

Theorem (B-Cartwright-D)
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Theorem (B-C-D)
The dual matroid is also that of a closed subgroup $X^\vee$ of G^n.

Colspace$(A)^\perp$ is a *left* subspace, but fortunately $Q \cong Q^{\mathrm{op}}$.

Dual valuations

Definition (Dress-Wenzel)
If $\mu : \binom{[n]}{d} \to \overline{\mathbb{R}}$ is a valuation, then $\mu' : \binom{[n]}{n-d} \to \overline{\mathbb{R}}$, $\mu'(I) = \mu(I^c)$ is the *dual valuation*.

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This notion is compatible with the dual of a linear matroid, but *not* with the construction of X' above: take $G = (K, +)$, $\mathbb{E} = K[F]$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & F \end{bmatrix} \longrightarrow A^{\perp} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & F & 0 & -1 \end{bmatrix} \longrightarrow A^{\vee} = \begin{bmatrix} 1 & 1 \\ 1 & F^{-1} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

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$\mu(14) + \mu(23) - \mu(13) - \mu(24) = 1 + 0 - 0 - 0 = 1$ but
$\mu^{\vee}(23) + \mu^{\vee}(14) - \mu^{\vee}(24) - \mu^{\vee}(13) = -1 + 0 - 0 - 0 = -1$

Theorem (B-C-D): The set of Lindström valuations of algebraic matroids is *not* closed under duality.

Proof sketch: via a universality construction of Evans-Hrushovski, we construct a matroid $M^\vee$ s.t. every algebraic realisation of $M^\vee$ is equivalent to one from a subgroup $X^\vee \subseteq G^n$ for some one-dimensional algebraic group G, but such that the Lindström valuation of X is not the dual to that of $X^\vee$. Then the dual of the Lindström valuation of X is not a Lindström valuation. $\square$

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Thank you!