Stabilisation in algebraic statistics

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Gaussian k-factor model

- $X_1, \ldots, X_n$ real, jointly Gaussian random variables, mean 0
- distribution determined by covariance matrix Σ
- suppose $n \gg k$ and $X_i = \sum_{j=1}^{k} \lambda_{ij} Y_j + \sigma_i Z_i$ where

$\underbrace{Y_1, \ldots, Y_k}_{\textit{factors}}, \underbrace{Z_1, \ldots, Z_n}_{\textit{noise}}$ independent, standard normal

$\rightsquigarrow \Sigma = \Lambda\Lambda^T + \mathrm{diag}(\sigma_i^2)_i$ rank k plus diagonal

- $F_{k,n} = \{\text{all such matrices}\} \subseteq \mathbb{R}^{n\times n}$

Question (DSS 07): Generators for the ideal in $\mathbb{R}[(\sigma_{ij})]$ of $F_{k,n}$?

Example

1. $\sigma_{ij} - \sigma_{ji}$ and
2. off-diagonal $(k+1)\times(k+1)$-minors of Σ
3. Ideal of $F_{2,5}$ gen by 1. and $\underbrace{\frac{1}{10}\sum_{\pi\in S_5} \mathrm{sgn}(\pi)\pi(\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{45}\sigma_{15})}_{\textit{pentad}}$

Stabilisation?

$F_{k,n+1} \to F_{k,n}$ by forgetting last row and col, and $F_{k,n}$ is S_n-stable

Question (DSS)
Is there $n_0 = n_0(k)$ s.t. $I(F_{k,n}) = \langle S_n I(F_{k,n_0})\rangle$ for $n \geq n_0$?

Partial answer
1. Yes for $k = 1$ with $n_0 = 4$ (2×2-minors).
2. Yes for $k = 2$ with $n_0 = 6$ (pentads and off-diagonal 3×3-minors generate $I(F_{2,n})$) [B-D]
3. Yes topologically for each k (no idea about n_0) [D]

FI-algebras

Definition An FI-algebra over K is a functor A from FI to (commutative, associative, unital) K-algebras. An ideal is an FI-submodule of A such that for each S, $I(S)$ is an ideal in $A(S)$.

General question

A an FI-algebra over K, $I \subseteq A$ an ideal of interest, is I f.g.?
(In k-factor model, $A(S) = \mathbb{R}[\sigma_{ij} | i, j \in S]$ and $I(S) = I(F_{k,S})$.)

not Noetherian!

FI-algebra $A \rightsquigarrow$ functor X from $\mathrm{FI}^{\mathrm{op}}$ to Top, $S \mapsto \mathrm{Hom}(A(S), K)$ with Zariski topology. Weaker question: is there a finite number of elements of $I(S)$'s whose ideal define $V(I) \subseteq X$?

The topological space defined by ideal $J \subseteq I(F_{k,\cdot})$ generated by the *off-diagonal* $(k + 1) \times (k + 1)$-minors is Noetherian.

Some results on FI-algebras

Theorem (C,A-H) The FI-algebra $R : S \mapsto K[S]$ is Noetherian.

Theorem (C, H-S) So are tensor powers $R^{\otimes d}$.

For any functor Y from FI to sets, can form the FI-algebra $K[Y] : S \mapsto K[Y_S]$.

Theorem (D-E-K-L) If Y is f.g. and $\phi : K[Y] \to R^{\otimes k}$ monomial, then $\ker \phi$ is f.g. and $\mathrm{im}\phi$ is Noetherian.

Unfortunately, $F_{k,n}$ was not parameterised by monomials (except for $k = 1$, forgetting diagonal), so this does not help there. But it does elsewhere!

Markov random fields

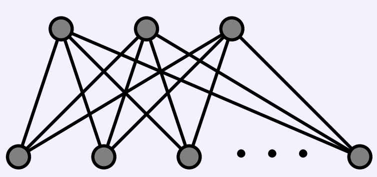
$\Gamma = (V, E)$ finite, simple, undirected graph
for each v have a finite set S_v of *states*
for each (max) clique $C \subseteq V$ have function $\phi_C : \prod_{v \in C} S_v \to \mathbb{R}_+$
$\rightsquigarrow$ a probability distribution on $\prod_{v \in V} S_v$ by $P(\mathbf{i}) = \prod_C \phi_C(\mathbf{i}|_C)$

Question
What are the polynomial relations among the $P(\mathbf{i})$ as the ϕ_C vary?

Examples

$P(i, j, k) = a_i b_j c_k$ (independence, rank-one tensors)

ideal generated by quadratic equations, *independently of the* $|S_v|$.



$K_{3,N}$ $P(i_1, i_2, i_3, j_1, \ldots, j_n) = \prod_{l=1}^{N} a_{i_1, j_l} b_{i_2, j_l} c_{i_3, j_l}$

Theorem (Rauh-Sullivant) for $K_{3,N}$, if all $|S_v| = 2$, ideal generated in degree ≤ 12, *independently of* N.

Theorem (Hillar-Sullivant)
Let $U \subseteq V$ be an independent set. Then the relations among the $P(\mathbf{i})$ stabilise as $|S_v| \to \infty$ for $u \in U$, while the S_v with $v \in V \setminus U$ remain fixed.

Proof
$\prod_{u\in U} S_u \mapsto K[p_{\mathbf{i}} | \mathbf{i} \in \prod_{v\in V} S_v]$ defines a f.g. FI^U-algebra
Consider the homomorphism $\phi : p_{\mathbf{i}} \mapsto P(\mathbf{i}) = \prod_C \phi_C(\mathbf{i}|_C)$

$= \prod_{C\cap U=\emptyset} \phi_C(\mathbf{i}|_C) \cdot \prod_{u\in U}(\prod_{C\ni u} \phi_C(\mathbf{i}|_C))$ (each $|C \cap U| \leq 1$)

constant stuff ↳ *only one index unbounded*

use the result about monomial maps. □

A second independent set theorem

Theorem (in progress, with Oosterhof, Rauh, Sullivant)
Let U be an independent set. Now keep all state set sizes equal, but repeatedly clone vertices in U, along with their sets S_u. Then the relations among the $P(\mathbf{i})$ stabilise up to symmetry.

For simplicity $|U| = 1$. Denote by W the set of clones, including the original vertex. (So W is independent in the new graph.)

$$P(\mathbf{i}) = \prod_{C \cap W = \emptyset} \phi_C(\mathbf{i}|_C) \cdot \prod_{w \in W} \left(\prod_{C \ni w} \phi_C(\mathbf{i}|_C)\right) \quad \text{(each } |C \cap W| \leq 1\text{)}$$

For each fixed value of $\mathbf{i}|_{V \setminus W}$ you see the entries of a rank-one tensor in $\mathbb{R}^{\otimes W}$. We'll prove a theorem about tuples of rank-one tensors.

An $\mathrm{FS}^{\mathrm{op}}$-algebras

Recall
FS is the category of finite maps with surjections

Theorem (Sam-Snowden)
f.g. $\mathrm{FS}^{\mathrm{op}}$-modules are Noetherian

Example of an $\mathrm{FS}^{\mathrm{op}}$-algebra
fix $n \in \mathbb{N}$, then surjection $S \to S'$ gives injection $[n]^{S'} \to [n]^S$ and hence algebra injection $K[y_\alpha \mid \alpha \in [n]^{S'}] \to K[y_\alpha \mid \alpha \in [n]^S]$. This is a f.g. $\mathrm{FS}^{\mathrm{op}}$ algebra T_n (for tensor), coord ring of $(K^n)^{\otimes S}$.

Theorem (Draisma-Kuttler)
For each fixed r, this algebra has a finitely generated $\mathrm{FS}^{\mathrm{op}}$-ideal whose zero set is the variety of border-rank-r tensors.

Rank-one tensors

In T_n, let I_n denote the ideal of the rank-one tensors.

Theorem (D-O-R-S)
$Q_n := T_n/I_n$ is a Noetherian $\mathrm{OS}^{\mathrm{op}}$-algebra; also for several copies.

Recall
OS has linearly ordered finite sets $[k]$ and morphisms
$f : [k] \to [l]$ such that $i < j$ implies $\max f^{-1}(i) < \max f^{-1}(j)$

$Q_n([l])$ is the monoid algebra of the additive monoid of matrices in $(\mathbb{Z}_{\geq 0})^{n \times l}$ with constant column sum.

Let $A \in (\mathbb{Z}_{\geq 0})^{n \times l}$ and $B \in (\mathbb{Z}_{\geq 0})^{n \times k}$ be such matrices. Call $A \leq B$ if $\exists$ ordered surjective $f : [k] \to [l]$ such that $b_{ij} \geq a_{if(j)}$ for all i, j.

Proposition
This is a w.p.o.

To each A associate the monomial ideal J_A in $K[x_1, \ldots, x_n]$ generated by the x^α with α running over the columns of A.

Theorem (Maclagan)
These monomial ideals are w.p.o. by reverse inclusion: in any sequence $J_1, J_2, \ldots \exists i < j : J_i \supseteq J_j$.

- Suppose there are bad sequences $A_1, A_2, \ldots$.
- Then these exist such that $J_{A_1} \supseteq J_{A_2} \supseteq \ldots$ (*).
- Take a minimal such bad sequence and write $A_i = (a_i | B_i)$.
- Find $i_1 < i_2 < \ldots$ such that $a_{i_1} \leq a_{i_2} \leq \ldots$ and moreover $J_{B_{i_1}} \supseteq J_{B_{i_2}} \supseteq \ldots$

Claim: $A_1, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$ is smaller bad sequence.

• suppose $A_i \leq B_j$, witnessed by $f : [l_j - 1] \to [l_i]$. Then $A_i \leq A_j$, witnessed by $g : [l_j] \to [l_i]$ with $g(m) = f(m-1), m > 1$ and $g(1) = r$ such that r-th column of $A_i \leq$ first column of A_j (exists by (*)).

• suppose that $B_i \leq B_j$, witnessed by $f : [l_j - 1] \to [l_i - 1]$. Then $A_i \leq A_j$ witnessed by $g : [l_j] \to [l_i]$ defined by $g(m) = f(m-1) + 1$ and $g(1) = 1$. □

The ideals in the second independent set theorem are $\mathrm{FS}^{\mathrm{op}}$-ideals in T_n (actually, a tensor product of copies), hence f.g.

There are many more graphical models, e.g. from phylogenetics, where stabilisation occurs!

Nonnegative rank

Definition
$M \in \mathbb{R}^{m\times n}_{\geq 0}$ has *nonnegative rank* $\leq r$ if $M = A \cdot B$ with $A \in \mathbb{R}^{m\times r}_{\geq 0}$ and $B \in \mathbb{R}^{r\times k}_{\geq 0}$.

Consider the boundary $B^{m\times n}_r$ of the set of matrices of nonnegative rank $\leq r$ in the variety of matrices of rank $\leq r$.

Theorem (Mond-Smith-v. Straten) for $r = 3$ this has 3 $S_m \times S_n$-orbits of components, independent of m, n.

(Quantifier-free description by Kubjas-Robeva-Sturmfels, and ideals by Eggermont-Horobet-Kubjas.)

Higher rank r? Ongoing work by Horobet-Chen.