Polynomial functors: topological Noetherianity and applications

Jan Draisma
Universität Bern
and TU Eindhoven

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Main Theorem

Let $X_1 \supseteq X_2 \supseteq \ldots$ closed subsets of a polynomial functor P.
Then $\exists i_0 : X_{i_0} = X_{i_0+1} = \ldots$

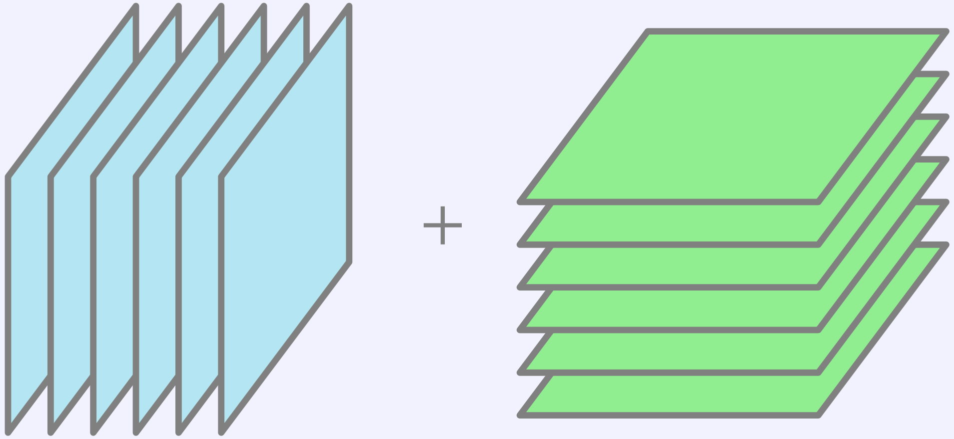
equivalently:

Main Theorem

For any closed subset X of P there is a fin-dim vector space U such that for each V, $X(V) \subseteq P(V)$ equals $\bigcap_{\varphi \in \mathrm{Hom}(V,U)} P(\varphi)^{-1} X(U)$.

Example

Let $P(V) = V^{\otimes 3}$ and $X(V) = \{v_1 \otimes A_{23} + v_2 \otimes A_{13}\}$, a subset of the tensors of *slice rank* ≤ 2.



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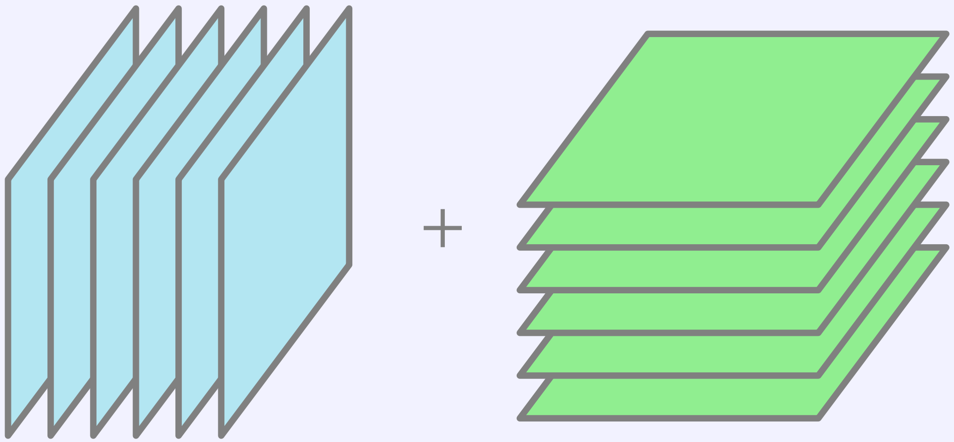

$X(V)$ is closed (notes by Tao-Sawin), and for $\varphi : V \to W$ have $P(\varphi)(X(V)) \subseteq X(W)$. Theorem implies $X(V)$ is defined by equations of bounded degree independent of V.

In fact, degrees 3 *and* 6 *suffice (Oosterhof)—set-theoretically.*

Question: also ideal-theoretically?

Polynomial functors

Setting
K an infinite field
Vec the category of finite-dimensional K-vector spaces

Definition
$P : \mathbf{Vec} \to \mathbf{Vec}$ is *polynomial* of degree $\leq d$ if $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}(P(V), P(W))$ is polynomial of degree $\leq d$ for all V, W.

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Examples

- $P(V) =$ a fixed U, of degree 0
- $P(V) = V$, of degree 1
- $P(V) = V^{\otimes d}$ with $P(\varphi)v_1 \otimes \cdots \otimes v_d = (\varphi v_1) \otimes \cdots \otimes (\varphi v_d)$
- $P(V) = S^d V = V^{\otimes d} / \langle \{v_1 \otimes \cdots \otimes v_d - v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}\} \rangle$
- char $K = p > 0$, $P(V) = S^p(V) / \langle f^p \mid f \in V \rangle_K$

Some background

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Lemma (Friedlander-Suslin) *(we'll use this)*
For fixed d and $U \in \mathbf{Vec}$ with $\dim U \geq d$ the map
$\{\text{poly functors of deg} \leq d\} \to \{\text{poly GL}(U)\text{-modules of deg} \leq d\}$,
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Theorem *(we won't use this)*
In char 0, Schur functors $S_\lambda(V) := \mathrm{Hom}_{S_d}(U_\lambda, V^{\otimes d}), \lambda \vdash d$ form a basis of the Abelian category of polynomial functors.

Quasi-inverse in Friedlander-Suslin's lemma

U a finite-dimensional vector space
$\mu : \mathrm{End}(U) \times \mathrm{End}(U) \to \mathrm{End}(U)$ multiplication
$\mu^* : K[\mathrm{End}(U)]_{\leq d} \to K[\mathrm{End}(U)]_{\leq d} \otimes K[\mathrm{End}(U)]_{\leq d}$ comult

$A := K[\mathrm{End}(U)]^*_{\leq d}$ associative algebra with multiplication $(x, y) \mapsto (x \otimes y) \circ \mu^*$; natural map $\mathrm{End}(U) \to A$ is algebra homomorphism.

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For a polynomial $\mathrm{GL}(U)$-module M of degree $\leq d$, get a map $A \otimes M \to M \rightsquigarrow$ bijection between polynomial $\mathrm{GL}(U)$-modules of degree $\leq d$ and A-modules.

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For a vector space V, $K[\mathrm{Hom}(U, V)]^*_{\leq d}$ is a *right* A-module.

Map back sends M to $V \mapsto K[\mathrm{Hom}(U, V)]^*_{\leq d} \otimes_A M$.

$P \rightsquigarrow$ a functor **Vec** $\to$ **Top**, where $P(V)$ has the Zariski topology

Definition

A **Vec**-*closed subset* X of P assigns to each V a closed subset $X(V) \subseteq P(V)$, such that for all $\varphi : V \to W$ the map $P(\varphi)$ maps $X(V)$ into $X(W)$. (*So* $\mathcal{I}(X(W))$ *pulls back into* $\mathcal{I}(X(V))$.)

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Examples

- Segre: $X(V) = \{v_1 \otimes \cdots \otimes v_d\} \subseteq V^{\otimes d}$
- Veronese: $X(V) = \{v^{\otimes d}\} \subseteq \Gamma^d V$
- joins $(X+Y)(V) := \overline{X(V) + Y(V)}$, tangential varieties, $\cup$, $\cap$, etc.

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Main Theorem **(Noetherianity for polynomial functors)**

In a polynomial functor P of finite degree, any chain $X_1 \supseteq X_2 \supseteq \dots$ of **Vec**-closed subsets stabilises: $X_{i_0} = X_{i_0+1} = \dots$ for some i_0.

Consequences and special cases

Known before

- degree 0 (finite-dimensional vector spaces, Hilbert, 1890)
- degree 1 (Cohen 1967)
- degree ≤ 2 (tuples of matrices, Eggermont 2014)
- for S^3V (cubics, Derksen-Eggermont-Snowden 2016)
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Slice rank revisited

$X_1(V) := \{T \in V^{\otimes d} \mid \exists i \in [d], v \in V, S \in \bigotimes_{j \neq i} V : T = v \otimes S\}$

$X_k := X_1 + \cdots + X_1$ tensors of *slice rank* $\leq k$

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The Main Theorem implies: $X_k(V)$ is defined set-theoretically by equations of bounded degree independent of V.

Twisted commutative algebras

$P \rightsquigarrow$ contravariant functor $V \mapsto K[P(V)]$ from **Vec** to K-algebras

Over $K = \mathbb{C}$, this is a *twisted commutative algebra* (Sam-Snowden). The Main Theorem implies that finitely generated tcas are topologically Noetherian. Ring-theoretically remains open.

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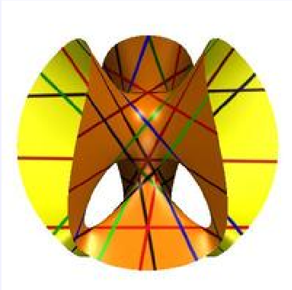
Variants of Stillman's conjecture [Erman-Sam-Snowden]

Let c be any natural number, and fix degrees $d_1, \ldots, d_k$. Then the number of codim-c linear subspaces of $\mathbb{P}^n$ contained in a projective variety defined by k polynomials of degrees $d_1, \ldots, d_k$ is either infinite or at most some number *which doesn't depend on n.*

(*Uses the main theorem for* $\bigoplus_{i=1}^{k} S^{d_i}(V)$.
A single non-rigid such space already counts as ∞.)
For $k = 1, d_1 = 3, c = 2$, N is at least 27:



Limits

For a polynomial functor P define $P_\infty := \lim_{\leftarrow n} P(K^n)$. This is dual to the countable-dimensional space $\lim_{n\to} P(K^n)^*$, whose elements we think of as coordinates on P_∞. We give P_∞ the Zariski-topology and write $K[P_\infty]$ for the coordinate ring.

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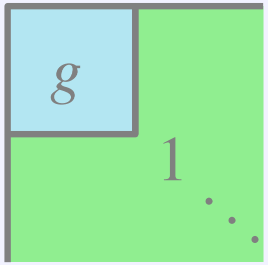
The group $\mathrm{GL}_\infty := \bigcup \mathrm{GL}_n$ acts on P_∞ and its space of coordinates, by homeomorphisms.



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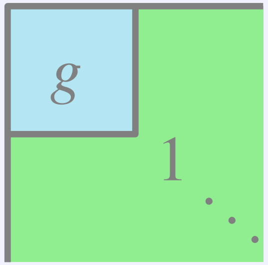

Example

- For $P(V) = V$, $P_\infty = K^{\mathbb{N}}$ with coordinates $x_1, x_2, \ldots$, and GL_∞ acts by matrix-vector multiplication.
- For $P(V) = V \otimes V$, $P_\infty = K^{\mathbb{N}\times\mathbb{N}}$ with coordinates x_{ij}, and GL_∞ acts by $(g, A) \mapsto gAg^T$.
- For $P(V) = S^3V$, P_∞ is the space of infinite cubics $a_{111}x_1^3 + a_{112}x_1^2x_2 + \cdots + a_{ijk}x_ix_jx_k$ and GL_∞ acts by coordinate substitutions.

Limits

Given a **Vec**-closed subset $X \subseteq P$, set $X_\infty := \lim_{\leftarrow n} X(K^n) \subseteq P_\infty$.

Exercise
The map $X \mapsto X_\infty$ is a bijection between **Vec**-closed subsets of P and GL_∞-stable closed subsets of P_∞.

(Show also that $Y \subseteq P_\infty$ closed and GL_∞-stable implies that the image of Y in $P(K^n)$ is closed.)

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Leads to another reformulation of the main theorem:

Theorem
For each GL_∞-stable, closed subset $Y \subseteq P_\infty$ there exists polynomials $f_1, \ldots, f_k \in K[P_\infty]$ such that $Y = \bigcap_{g \in \mathrm{GL}_\infty} \bigcap_i V(g f_i)$.

Warm-up: symmetric matrices

$P(V) :=$ {symmetric tensors in $V \otimes V$} $= \Gamma^2 V$, assume char $K = 0$

P_∞ = {infinite-by-infinite symmetric matrices} with $K[P_\infty] = K[x_{ij}]/\langle (x_{ij} - x_{ji})_{i,j} \rangle$.

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Let be $X \subseteq P_\infty$ closed and GL_∞-stable and $f \in \mathcal{I}(X) \cap K[x_{ij} | i, j \leq n]/(x_{ij} - x_{ji})$ homogeneous of minimal degree vanishing on X. After acting with GL_n, may assume that x_{nn} appears in f.

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Compute action of $E_{n,n+1} \in \mathfrak{gl}_\infty$ by $(1 + tE_{n,n+1})A(1 + tE_{n,n+1})^T$ mod $t^2 \rightsquigarrow$ action is by $\sum_j x_{n+1,j} \frac{\partial}{\partial x_{n,j}} + \sum_i x_{i,n+1} \frac{\partial}{\partial x_{i,n}}$ on $K[P_\infty]$. Now $\tilde{f} := E_{n,n+1}^2 f$ also vanishes on X. Have $\tilde{f} = x_{n+1,n+1} h + r$ where $h \in K[x_{ij} \mid i, j \leq n]$ and r does not involve $x_{n+1,n+1}$.

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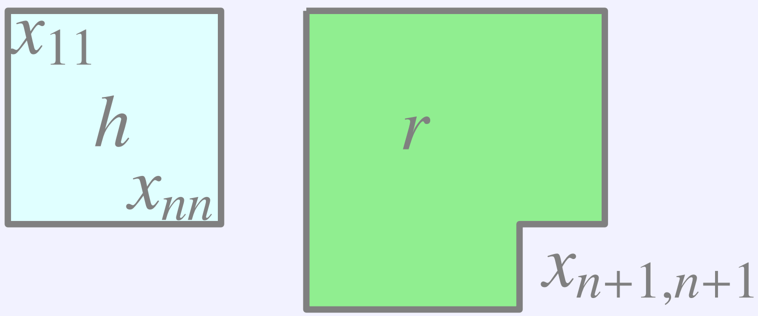
Have found $\tilde{f} = x_{n+1,n+1}h + r \in \mathcal{I}(X)$ with supports of h and r as follows:



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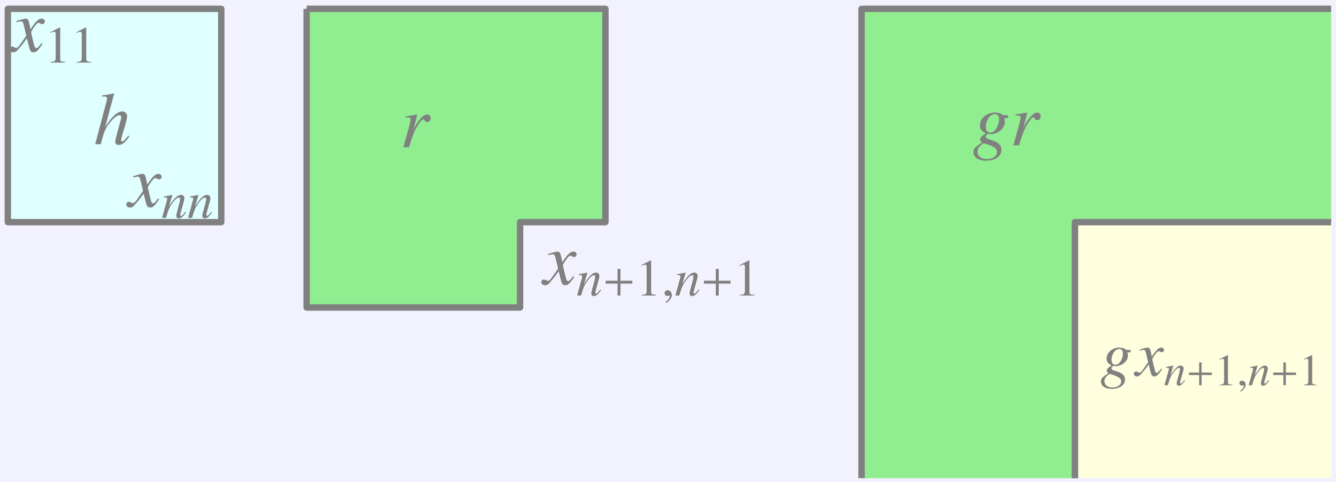

For any $g \in \mathrm{GL}_\infty$ fixing $e_1, \ldots, e_n$ find $g\tilde{f} = (gx_{n+1,n+1})h + gr \in \mathcal{I}(X)$. *So on the locus in* X *where* $h \neq 0$, *we can express the yellow coordinates in the green coordinates.*

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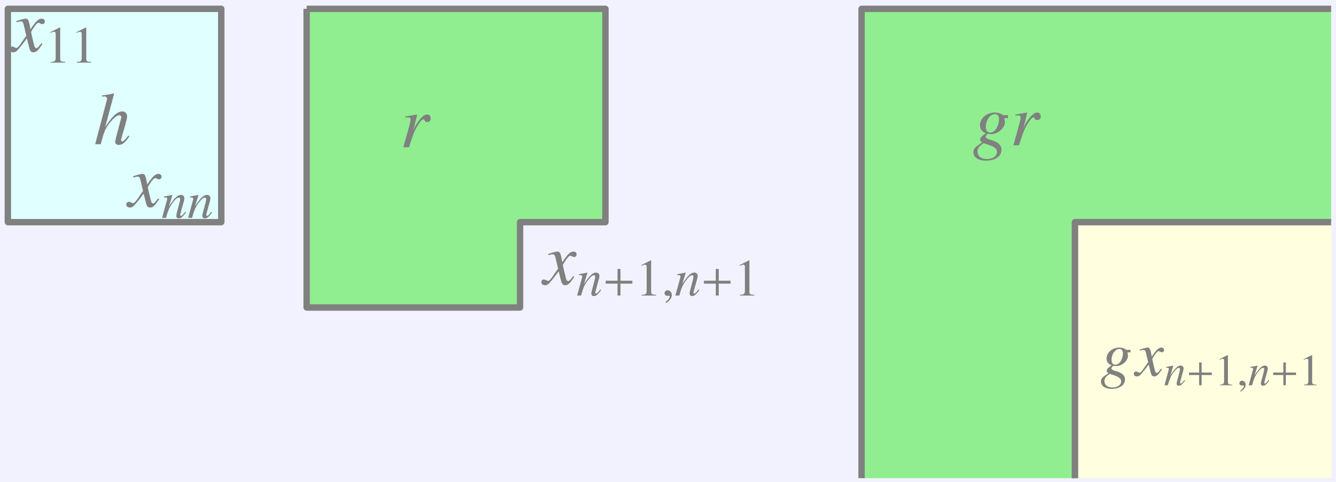

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Decompose $X = Y \sqcup Z$ where $Y = X \cap \bigcap_{g \in \mathrm{GL}_\infty} V(gh)$ and $Z = \bigcup_{g \in GL_\infty} g(X \setminus V(h))$.

Warm-up: symmetric matrices

Now:

• on Y, a lower-degree polynomial vanishes (namely, h), so it is Noetherian by induction;

• $X \setminus V(h)$ is isomorphic to a subset in the limit of a smaller polynomial functor, namely, $V \mapsto (\Gamma^2 K^n) \oplus (K^n \otimes V) \oplus (V \otimes K^n)$ with the green coordinates. So it is Noetherian by induction.

We conclude that X is Noetherian. $\square$

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Note that the smaller functor is obtained from P by taking the quotient of $P(K^n \oplus V) = (\Gamma^2 K^n) \oplus (K^n \otimes V) \oplus (V \oplus K^n) \oplus (\Gamma^2 V)$ by the top-degree part $\Gamma^2 V$.

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The general case will be similar; we'll assume $\deg P > 0$ as degree-zero case follows from Hilbert's basis theorem.

Proof ingredients

The shift functor

U a fixed vector space $\rightsquigarrow \mathrm{Sh}_U : \mathbf{Vec} \to \mathbf{Vec}, V \mapsto U \oplus V$

Exercise

If P is a polynomial functor of degree d, then $P \circ \mathrm{Sh}_U$ is also a polynomial functor of degree d, and $P_d \cong (P \circ \mathrm{Sh}_U)_d$.

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Example
$\mathrm{Sh}_U(V) = S^d(U \oplus V) = \bigoplus_{e=0}^{d} S^{d-e}U \otimes S^eV = S^dV + \cdots$

Note that $(P \circ S_U)_e$ is larger than P_e for $e < d$.

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Remark

In the proof for S^2, we used $U = K^n$.

A lexicographic order
Define $Q \prec P$ if $Q \ncong P$ and for the largest e with $Q_e \ncong P_e$ the former is a homomorphic image of the latter.

By [FS], a well-founded order on finite-degree polynomial functors. Do induction, and assume the theorem holds for all $Q \prec P$.

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For $X \subseteq P$ let X_Q be the closure of the image in Q. Think of X as a variety over X_Q. Accordingly, $\mathcal{I}_X(V)$ is the ideal of X in $K[\pi(V)^{-1}(X_Q(V))] \cong K[X_Q(V)] \otimes K[R(V)]$ (non-canonically).

Another well-founded order

Define $\delta_X \in \{1, 2, \ldots, \infty\}$ as the minimal degree of a nonzero polynomial in $\mathcal{I}_X(V)$ over all V. $(\delta_X > 0?)$

For $X, Y \subseteq P$ say $X > Y$ if $X_Q \supsetneq Y_Q$ or $X_Q = Y_Q$ and $\delta_X > \delta_Y$. As $Q \prec P$, Q is Noetherian by the induction hypothesis, so this is a well-founded order.

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Second induction hypothesis

All $Y < X$ are Noetherian.

Now if $\delta_X = \infty$, X Noetherian. So assume $\delta_X \in \mathbb{Z}_{\geq 1}$.

Another well-founded order
Define $\delta_X \in \{1, 2, \ldots, \infty\}$ as the minimal degree of a nonzero polynomial in $\mathcal{I}_X(V)$ over all V. $(\delta_X > 0?)$

For $X, Y \subseteq P$ say $X > Y$ if $X_Q \supsetneq Y_Q$ or $X_Q = Y_Q$ and $\delta_X > \delta_Y$. As $Q \prec P$, Q is Noetherian by the induction hypothesis, so this is a well-founded order.

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Let $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ chain of **Vec**-closed subsets of X.

Proof steps (char zero)

- Take $f \in \mathcal{I}_X(U)$ nonzero, homog of degree δ_X, $\dim U$ minimal
- Pick an $r_0 \in R(U)$ such that $h := \partial_{r_0} f$ is nonzero.

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- Define $Y(V) := \{q \in X(V) \mid \forall \varphi : V \to U, h(P(\varphi)q) = 0\}$.
- Then either $Y_Q \subsetneq X_Q$ or else $Y_Q = X_Q$ and $\delta_Y \leq \deg h < \delta_X$, so Y is Noetherian, so $(Y \cap X_i)_i$ stabilises.

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- Define $Z(V) := X(V) \setminus Y(V)$. Goal: $(Z \cap X_i)_i$ also stabilises.
- Define $X' := X \circ \mathrm{Sh}_U, P' := P \circ \mathrm{Sh}_U$; then $Q' := P'/R \prec P$, so Q' is Noetherian.

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- Hence $(Z' \cap (X_i \circ \mathrm{Sh}_U))_i$ stabilises, and hence $(Z \cap X_i)_i$ stabilises. $\square$

Theorem
Every polynomial functor P is topologically Noetherian.

- For $X \subseteq P$ find an equation f of smallest degree vanishing on it
- Set $h := \partial_{r_0} f$ for some r_0 in an irreducible subfunctor R in the top-degree part of P.
- Split $X = Y \cup Z$, where Y is defined by $h = 0$ and Z embeds in a smaller polynomial functor Q'.
- Y is Noetherian since $\deg h < \deg f$ and Z because it sits in Q'.

Remarks

• In positive characteristic, might have $\partial_{r_0} f = 0$ for all $r_0 \in R(U)$. In that case, f is a polynomial in the p^e-th powers of the coordinates on $R(U)$, with coefficients from $K[X_Q(U)]$. Take e maximal with this property $\rightsquigarrow$ argument doesn't change much.

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• In characteristic zero, can find a morphism $\iota : B' \to P$ such that $\iota \circ \pi'_Q$ is the identity on Z' (ongoing work with Bik-Eggermont-Snowden); see Eggermont's talk on strength.

Outlook

Will discuss:

- vary the functor, e.g. $\bigwedge^p V$ with varying p
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Probably much easier:

Is there a $\mathbb{Z}$-version of the topological results discussed here?

This is an example of varying the polynomial functor.

Definition
A *Plücker variety* is a sequence $X_p \subseteq \bigwedge^p$, $p = 0, 1, 2, 3, \ldots$ of **Vec**-closed subsets such that for all $V \in \mathbf{Vec}$ and $x \in V^*$, contraction with v maps $X_p(V)$ into $X_{p-1}(V)$.

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Examples

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Theorem (D-Eggermont)

Every Plücker variety is defined set-theoretically by equations of bounded degree (uniform in p and V).

Proof sketch: take p_0 and U and a minimal-degree f on $\bigwedge^{p_0} U$ that vanishes on X_{p_0}. Using the same technique as in the proof for polynomial functors, find a derivative h, and let $Y \subseteq X$ be the Plücker subvariety defined by $h = 0$. Prove that any Plücker subvariety $X' \subseteq X$ is uniquely determined by $X' \cap Y, X'_0, \ldots, X'_{p_0}$.

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Remark: Robert Laudone has a much better theorem for secants of Grassmannians, which implies that their ideals and higher syzygies are generated in bounded degree.

Stillman's conjecture

Theorem (Ananyan-Hochster, Erman-Sam-Snowden)
Fix $d_1, \ldots, d_k$. Then there exists an N such that for any field K and any n and any homogeneous $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_k$ the projective dimension of $\langle f_1, \ldots, f_k \rangle$ is at most N.

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Long history: arbitrary number of quadrics (A-H), three cubics (Engheta), ...

Theorem (D-Łasoń-Leykin)
There exists a finite algorithm that on input $d_1, \ldots, d_k$ outputs all possible *generic grevlex initial ideals* of such ideals.

• implies the previous theorem, since projective dimension is preserved under passing to (generic) grevlex initial ideals
• proof is similar in spirit to, and uses results of, E-S-S.

Grevlex initial ideals

Construction (generic initial ideals)

• Let $<$ be a monomial order on $K[x_1, \ldots, x_n]$ and let $I \subseteq K[x_1, \ldots, x_n]$ a homogeneous ideal.

• Write $\mathrm{in}_< I$ for the ideal spanned by the leading (largest) monomials of elements of I.

• The set of $g \in \mathrm{GL}_n$ where $\mathrm{in}_< gI$ is a fixed ideal is a constructible set, and finitely many of these partition GL_n.

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Definition (grevlex)

We use $<$ on $K[x_1,\ldots,x_n]$ defined by $x^\alpha < x^\beta$ if $\sum_i \alpha_i < \sum_i \beta_i$ or $=$ and then the last nonzero element of $\alpha - \beta$ is *negative*.

So $x_1^3 > x_1^2 x_2 > x_1 x_2^2 > x_2^3 > x_1^2 x_3 > x_1 x_2 x_3 > x_2^2 x_3 > x_1 x_3^2 > \ldots$

A big polynomial ring

Definition

$R = R_K$ = the ring of bounded-degree series in $x_1, x_2, x_3, \ldots$

So a homogeneous cubic element of R looks like

$a_{111}x_1^3 + a_{112}x_1^2x_2 + a_{122}x_1x_2^2 + a_{222}x_2^3 + a_{113}x_1^2x_3 + \cdots$

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Theorem (Erman-Sam-Snowden)
Assume K is perfect. Then R is isomorphic to a polynomial ring (in uncountably many variables).

Let $f_1, \ldots, f_k \in R$ be homogeneous. Define
$\pi_n : R \to K[x_1, \ldots, x_n]$ the projection.

Proposition (using Erman-Sam-Snowden)
The map $\mathrm{Syz}(\pi_{n+1}(f_1), \ldots, \pi_{n+1}(f_k)) \to \mathrm{Syz}(\pi_n(f_1), \ldots, \pi_n(f_k))$ is surjective for $n \gg 0$.

Proposition
Let $f_1, \ldots, f_k \in R$. Then $\langle f_1, \ldots, f_k \rangle$ has a finite grevlex Gröbner basis.

(Run Buchberger's algorithm for the truncations $\pi_n(f_1), \ldots, \pi_n(f_k)$ dragging along but ignoring the terms divisible by x_i with $i > n$ until the computation is finished, and then increase n by one. By the proposition on syzygies, this terminates.)

Proof sketch of the theorem on generic initial ideals

• Start running the Buchberger algorithm, but now regard the coefficients of the f_i as indeterminates (coordinates on $S_\infty^{d_1} \oplus \cdots \oplus S_\infty^{d_k}$).

• Whenever you need to decide if some polynomial p in these coefficients is zero, branch the computation into:

• An *open branch* where p is assumed nonzero (localise);

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• An *open branch* where p is assumed nonzero (localise);
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• By Noetherianity of $S^{d_1} \oplus \cdots \oplus S^{d_k}$, in each path in this computation, only finitely many closed branches are followed.
• Hence all paths eventually only follow open branches. But then they are performing a Gröbner basis computation in R_L for some extension field L of K, hence they terminate.

Hence the entire tree of computations is finite. □

Summarising

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Thank you!