Amoeba dimensions

Jan Draisma (Bern/Eindhoven)   Oberwolfach, March 2023

joint with Eggleston, Pendavingh, Rau, and Yuen

Amoebas

$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$
$X \subseteq (\mathbb{C}^*)^n$ irreducible closed subvariety

Definition
$\mathcal{A}(X) := \mathrm{Log}(X)$ *amoeba* of X (essentially semi-algebraic)
$\mathrm{Trop}(X) := \lim_{t\to\infty} \frac{1}{t}\mathrm{Log}(X)$ *tropicalisation* of X

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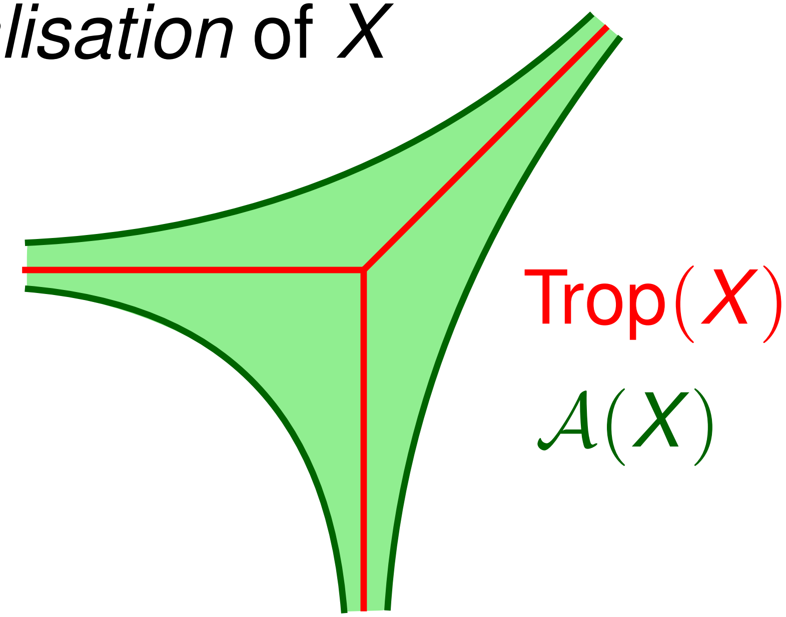

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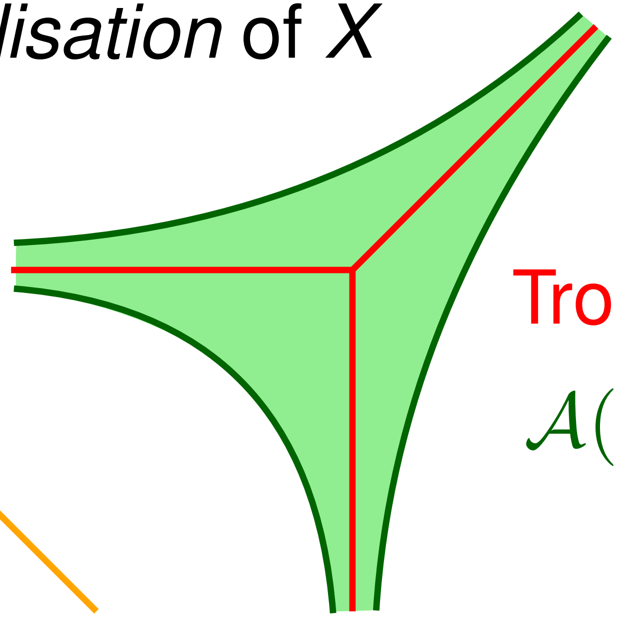

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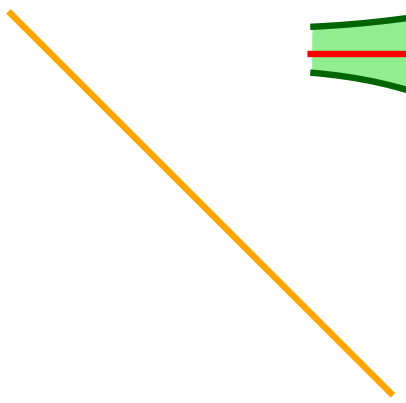

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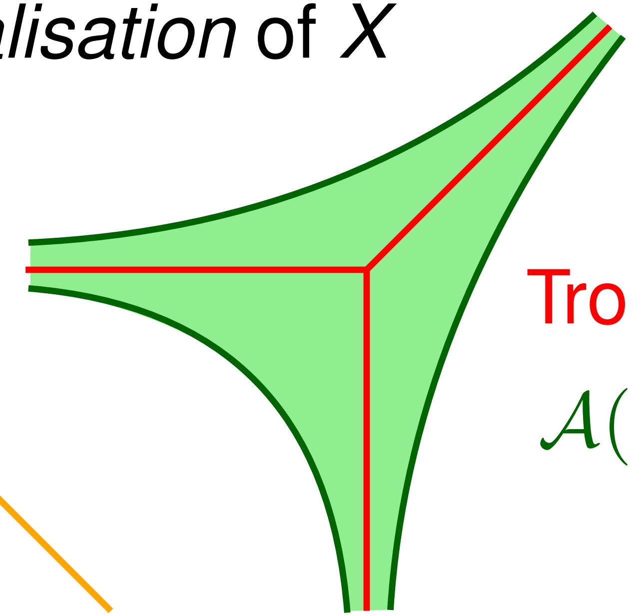

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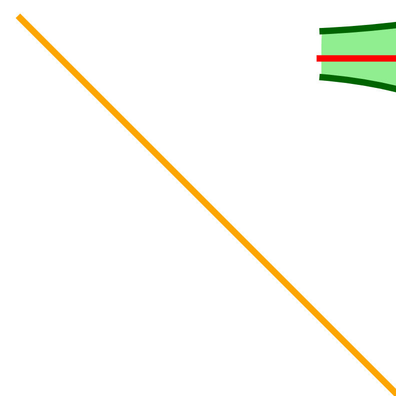

Example 3
$T \subseteq (\mathbb{C}^*)^n$ a k-dimensional torus $\rightsquigarrow \mathcal{A}(T) = \mathrm{Trop}(T)$ a k-dim real subspace of $\mathbb{R}^n$ spanned by rational vectors.

Amoeba dimensions?

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Theorem (Bergman, Bieri-Groves, . . .)
$\mathrm{Trop}(X)$ is a finite union of $\dim_{\mathbb{C}} X$-dimensional polyhedra.

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Observation (Nisse-Sottile)
X invariant under k-dimensional subtorus $T \subseteq (\mathbb{C}^*)^n$
$\rightsquigarrow X/T \subseteq (\mathbb{C}^*)^n/T \cong (\mathbb{C}^*)^{n-k}$
$\rightsquigarrow \mathcal{A}(X) \to \mathcal{A}(X/T)$ has fibres $\cong \mathcal{A}(T)$ of dimension k
$\rightsquigarrow \dim_{\mathbb{R}}(\mathcal{A}(X)) = \dim_{\mathbb{R}}(\mathcal{A}(X/T)) + 1 \cdot \dim_{\mathbb{C}}(T)$

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Theorem (Nisse-Sottile)
$\dim_{\mathbb{R}} \mathcal{A}(X) \geq 1 \cdot \dim_{\mathbb{C}} X$ with $=$ iff X is a torus orbit.

Dimension drop and near torus actions

Example 4

$X \subseteq (\mathbb{C}^*)^n$ of dimension $> n/2$

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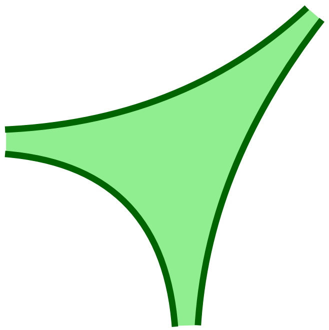
Definition (Nisse-Sottile, D-Rau-Yuen)
If $T \subseteq (\mathbb{C}^*)^n$ subtorus such that $\mathcal{A}(T) \subseteq T_p\mathcal{A}(X)$ for almost all $p \in \mathcal{A}(X)$, then we say T *nearly acts* on X.



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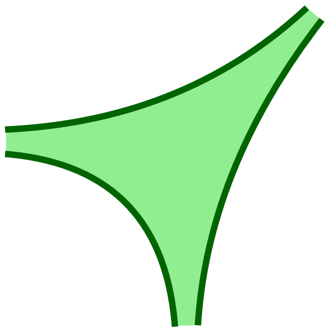

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Proposition (D-Rau-Yuen)

$\dim_{\mathbb{R}}(\mathcal{A}(X)) < 2 \cdot \dim_{\mathbb{C}}(X) \Rightarrow$ some T, $\dim_{\mathbb{C}}(T) > 0$, nearly acts on X. And then $\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{R}}(\mathcal{A}(\overline{T \cdot X}))$.

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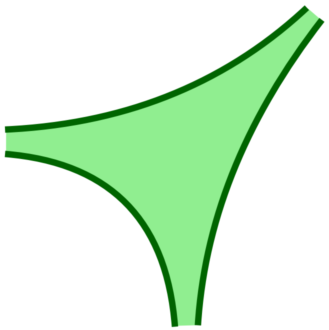

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Corollary
$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2 \cdot \dim_{\mathbb{R}}(\mathrm{Trop}(X) + T) - \dim_{\mathbb{R}}(T) \mid$
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Observation
If $X = V \cap (\mathbb{C}^*)^n$, where $V \subseteq \mathbb{C}^n$ is a $\mathbb{C}$-linear subspace
$\rightsquigarrow$ Trop$(X) =$ Bergman fan $B(M_V)$ of matroid M_V on $[n]$.
Can $\dim_{\mathbb{R}} \mathcal{A}(X)$ be computed efficiently in this case?

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Remarks

- $\leq$ follows from $V \subseteq \prod_i V_{P_i}$ so that $\mathcal{A}(X) \subseteq \prod_i \mathcal{A}(X_{P_i})$
- Proof of $\geq$ is independent of DRY.
- $(*)$ defines a matroid M', $1 \cdot \mathrm{rk}(M) \leq \mathrm{rk}(M') \leq 2 \cdot \mathrm{rk}(M)$.

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Open: Does this hold for *all* loop-free matroids on $[n]$?
$(*) = \min\{2 \cdot \dim_{\mathbb{R}}(B(M) + T) - \dim_{\mathbb{R}}(T) \mid T \subseteq \mathbb{R}^n \text{ subspace spanned by rational vectors}\}$

Proof of first part

- W.l.o.g. $\mathbb{1} := (1, \ldots, 1) \in X$ has $d_{\mathbb{1}}\mathrm{Log} : T_{\mathbb{1}}X = V \to \mathbb{R}^n$ has maximal rank. If $2 \cdot \dim_{\mathbb{C}} V - 1$, done.

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$= w \mapsto (v \mapsto$ coefficient of ϵ in $\Re(v/(\mathbb{1} + \epsilon w)))$
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- Near $\mathbb{1}$, $d_p\mathrm{Log}$ has constant rank $\rightsquigarrow$ $v \mapsto \Re(-vw)$ maps $\ker(d_{\mathbb{1}}\mathrm{Log}) = V \cap i\mathbb{R}^n$ into $\mathrm{im}(d_{\mathbb{1}}\mathrm{Log})$ for all $w \in V$. So $(V \cap i\mathbb{R}^n) \cdot V \subseteq \mathrm{im}(d_{\mathbb{1}}\mathrm{Log}) + i\mathbb{R}^n \subseteq V + i\mathbb{R}^n$ and $(V \cap \mathbb{R}^n) \cdot V \subseteq V + \mathbb{R}^n$. So $(V \cap \mathbb{R}^n) \cdot (V + \mathbb{R}^n) \subseteq V + \mathbb{R}^n$

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- $V + \mathbb{R}^n$ closed under multiplying with powers of $u \in (V \cap [0,1]^n) \setminus \mathbb{R}\mathbb{1} \rightsquigarrow$ gives a split $[n] = P_1 \sqcup P_2$; do induction. $\square$

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Thank you!

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