LIE ALGEBRAS OF VECTOR FIELDS

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Preface

The topic of this thesis dates back to Sophus Lie, who introduced Lie algebras of vector fields for his study of differential equations. Lie worked especially hard to classify the finite-dimensional transitive Lie algebras up to coordinate changes; he completed the case of two variables, and published many classes of transitive Lie algebras in three variables. This thesis presents a variety of results on finite-dimensional, often transitive, Lie algebras of vector fields in any number of variables, without trying to perform the hopeless job of classifying them.

The Realization Theorem of Guillemin and Sternberg translates transitive Lie algebras in n variables with formal power series coefficients to pairs $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is an ideal-free subalgebra of codimension n in the Lie algebra $\mathfrak{g}$. Blattner's proof of their theorem leads to an explicit formula for a realization of such an abstract pair. This Realization Formula and its consequences are treated in Chapter 2.

Once we are able to compute realizations with formal power series coefficients, the question arises whether smaller algebras of coefficients suffice. For example, in Section 2.3 we prove that our Realization Formula always leads to absolutely convergent coefficients. More restrictively, Lie conjectured that each finite-dimensional transitive Lie algebra has a conjugate in which the coefficients are polynomials in the variables x_i and the exponential functions $\exp(\lambda x_i)$ for some constants λ; in Chapter 2 we prove this conjecture in several special cases, in particular the case where $n \leq 3$. However, the arguments needed for $n = 3$ are already so *ad hoc* that I am tempted to disbelieve the conjecture for n sufficiently large. In analogy, section 2.7 treats the question of whether any transitive Lie algebra has a conjugate with coefficients that are algebraic functions. This is true for $n = 1$, but we show that it is not true for $n \geq 2$. It seems to me that a similar trick should eventually settle Lie's conjecture. The main results of Chapter 2 are Theorem 2.3.1, which presents the explicit realization formula mentioned above; Theorem 2.3.4, stating the equivalence of classification of formal transitive Lie algebras with classification of convergent ones; Theorem 2.2.4, which proves a special case of Lie's conjecture; and Theorem 2.2.5, which proves Lie's conjecture in codimensions up to three.

Chapter 3 deals with a particularly nice class of transitive Lie algebras, namely those corresponding to pairs $(\mathfrak{g}, \mathfrak{k})$ as before, with the additional condition that $\mathfrak{k}$ be maximal in $\mathfrak{g}$. There is a natural notion of inclusion of one such pair in another one of the same codimension, and this chapter investigates the existence and non-existence of inclusions between such pairs. Not surprisingly, these results have meanings in terms of realizations, and in terms of embeddings between homogeneous spaces in the algebraic group setting. The main results of Chapter 3 are Theorem 3.4.4, showing that there are only few inclusions among so-called simple-parabolic pairs; and Theorem 3.5.9,

Conjecture 3.5.12 and Theorem 3.5.15, which are similar non-existence statements on embeddings into simple-parabolic pairs.

Polynomial coefficients appear in the context of Blattner's realization, see Section 2.4, but also in the setting of algebraic group actions, which are the topic of Chapter 4. Indeed, if an affine algebraic group G acts on an algebraic variety V, then there is a natural action of the Lie algebra $\mathfrak{g}$ of G by derivations on the sheaf of regular functions on V. In particular, on an open subset U of V that is isomorphic to an affine space, this yields a realization of $\mathfrak{g}$ by means of polynomial vector fields. More generally, if U is an open affine subset of V, then we have an action of $\mathfrak{g}$ by derivations on the affine algebra $K[U]$. This raises a natural inverse question: given a Lie algebra $\mathfrak{g}$ acting by derivations on an affine algebra $K[U]$, does this action correspond to an algebraic group action on U, or on an algebraic variety containing U as an open dense subset? This question, too, is treated in Chapter 4. The main results of this chapter are Theorems 4.4.2 and 4.5.4, presenting necessary and sufficient conditions, in the locally finite and the general case respectively, for a Lie algebra of vector fields to integrate to an algebraic group action; and their consequences, Theorems 4.1.1, 4.1.2 and 4.1.3.

Chapter 5 deals with representation theory connected with certain transitive Lie algebras, namely the simple graded Lie algebras of depth 1. Both Blattner's realization and the algebraic group argument show that such a Lie algebra $\mathfrak{g}$ has a graded transitive embedding into the Lie algebra $\mathfrak{D}$ of derivations on a polynomial algebra, and we investigate the structure of $\mathfrak{D}$ as a $\mathfrak{g}$-module. This structure is closely related to the beautiful theory of (generalized) Verma modules and the category $\mathcal{O}$. The main results of this chapter are Theorem 5.4.2 and Conjectures 5.4.4 and 5.4.5, which describe the irreducible composition factors of $\mathfrak{D}$.

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CHAPTER 1

Introduction

The intention of this introduction is threefold. First, it is meant to present the subject of this thesis and to expose the relations between the rather diverse themes of the subsequent chapters. Second, this chapter will give a brief overview of the history of the area, touching, however, only those contributions that are indispensable to appreciate the general theme of the present work. Third, this introduction serves its name well in introducing some terminology that is used throughout the thesis at hand.

When describing the history of a mathematical subject, it is convenient to have the modern terminology of the area at one's avail, which is after all the product of an evolution in which suitability to explain known phenomena and to explore new ones is the main selection rule. Therefore, I choose to start with Guillemin and Sternberg's description of transitive differential geometry, before describing the works of Lie, Morozov, and Dynkin.

1.1. Formal Transitive Differential Geometry

There is a close analogy between Guillemin and Sternberg's theory of transitive differential geometry and the notion of transitive group actions, so we first review the latter briefly.

A pair (G, H) of groups, where H is a subgroup of G, defines the homogeneous space G/H with base point eH, on which G acts transitively. The action is faithful (or 'effective') if and only if H contains no non-trivial normal subgroups of G, and primitive if and only if H is a maximal subgroup of G. Conversely, if G acts transitively on a set V, and if $p_0 \in V$, then V can be identified, as a set with G-action, with G/G_{p_0}, where G_{p_0} is the stabilizer of p_0 in G.

There is yet another point of view: if G acts on a set V (not necessarily transitively), and if K is a field, then G acts on the set K^V of K-valued functions on V by $gf(p) = f(g^{-1}p)$ for all $g \in G, f \in K^V$ and $p \in V$. Pointwise multiplication and addition defines the structure of a K-algebra on K^V, and for all $g \in G$ the map $f \mapsto gf$ is an automorphism of this algebra. Conversely, under additional assumptions on K and V, and when restricting to some subalgebra R of K^V, one may recover V from R as the set of all maximal ideals of the latter. This is the case, for example, when V is finite and $R = K^V$, or when K is algebraically closed, V is an affine algebraic variety over K, and $R = K[V]$ is the algebra of regular functions on V ([**5**], AG.5). If, in addition, R is invariant under the action of G defined above, then G permutes the maximal ideals of R, and we thus recover the action of G on V. In the case where V is finite and $R = K^V$, this condition is automatically fulfilled; in the second example, the condition that G be an affine algebraic group over K and that the action of G on V be

a morphism $G \times V \to V$ of algebraic varieties, is sufficient for invariance of $K[V]$ under G. Somewhat imprecisely, we thus find a correspondence between pairs (G, H) where H is a subgroup of the group G, and triples (G, R, I) where G acts by automorphisms on the commutative algebra R permuting the maximal ideals transitively, and where I is a maximal ideal in R.

Formal transitive differential geometry constructs a similar correspondence between pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{g}$, and—in some sense transitive—actions of $\mathfrak{g}$ by *derivations* on some algebra. To introduce the latter, let A be an associative (not necessarily commutative) algebra over a field K and let M be a A-bimodule. Then $\mathrm{Der}_K(A, M)$ is the space of all K-linear maps $X : A \to M$ satisfying *Leibniz' rule*: $X(ab) = X(a)b + aX(b)$ for all $a, b \in A$. Note that, if A is commutative, then we can turn any left A-module M into an A-bimodule by setting $ma = am$ for all $a \in A$ and $m \in M$. The elements of $\mathrm{Der}_K(A, M)$ are called *derivations on A with values in M*. In particular, we write $\mathrm{Der}_K(A)$ for $\mathrm{Der}_K(A, A)$, where the left action and the right action of A on itself are left and right multiplication, respectively. The elements of $\mathrm{Der}_K(A)$ are simply called *derivations on A*. They are also defined for non-associative K-algebras, and $\mathrm{Der}_K(A)$ is a Lie algebra with respect to the commutator. If A is a finitely generated commutative integral domain, then derivations of A may be interpreted as *vector fields* on the affine algebraic variety $\mathrm{Spec}_K(A)$.

Let K be a field of characteristic zero, let n be a positive integer, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ denote a list of indeterminates. Denote by $K[[x_1, \ldots, x_n]] = K[[\mathbf{x}]]$ the K-algebra of formal power series in the x_i with coefficients from K. Its elements are written as
$$f = \sum_{\mathbf{m} \in \mathbb{N}^n} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}},$$
where $\mathbf{x}^{\mathbf{m}} := \prod_{i=1}^n x_i^{m_i}$ for $\mathbf{m} = (m_1, \ldots, m_n)$ a multi-index. The algebra $K[[\mathbf{x}]]$ is local with maximal ideal
$$M := \{f \in K[[\mathbf{x}]] \mid f(0) = 0\}.$$
The powers of M are of the form
$$M^d = \left\{ \sum_{\mathbf{m} \in \mathbb{N}^n} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \mid c_{\mathbf{m}} = 0 \text{ for } |\mathbf{m}| < d \right\},$$
where $|\mathbf{m}| := m_1 + \ldots + m_n$ is the *total degree* of $\mathbf{m}$. They form a fundamental system of neighbourhoods of 0 in the so-called *M-adic topology*. Automorphisms of $K[[\mathbf{x}]]$ leave M, whence every power of M, invariant, and are therefore continuous in the M-adic topology; they are also called *coordinate changes*. Similarly, derivations of $K[[\mathbf{x}]]$ map M^d into M^{d-1}, and are also continuous. Therefore, elements of $\mathrm{Aut}_K(K[[\mathbf{x}]])$ and $\mathrm{Der}_K(K[[\mathbf{x}]])$ are determined by their values on the x_i; this implies that elements of the latter space can be written as
$$\sum_{i=1}^n f_i \partial_i,$$
where the f_i are elements of $K[[\mathbf{x}]]$ and the derivations ∂_i are defined by
$$\partial_i \left(\sum_{\mathbf{m} \in \mathbb{N}^n} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \right) := \sum_{\mathbf{m} \in \mathbb{N}^n} c_{\mathbf{m}+\mathbf{e}_i}(m_i + 1)\mathbf{x}^{\mathbf{m}},$$

in which $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{N}^n$. The formal power series f_i are called the *coefficients* of the derivation. The commutator turns $\mathrm{Der}_K(K[[\mathbf{x}]])$ into a Lie algebra over K, which we shall denote by $\hat{\mathfrak{D}}^{(n)}$ if we want to stress the number of variables, and by $\hat{\mathfrak{D}}$ otherwise. Its elements are also called (formal) *vector fields*.

The Lie algebra $\hat{\mathfrak{D}}$ has a natural filtration

$$\hat{\mathfrak{D}} = \hat{\mathfrak{D}}_{-1} \supset \hat{\mathfrak{D}}_0 \supset \hat{\mathfrak{D}}_1 \supset \hat{\mathfrak{D}}_2 \supset \ldots$$

where $\hat{\mathfrak{D}}_d$ consists of those derivations all of whose coefficients are in M^{d+1}, or, equivalently,

$$\hat{\mathfrak{D}}_d := \{X \in \hat{\mathfrak{D}} \mid X(M^e) \subseteq M^{d+e} \text{ for all } e \in \mathbb{N}\}.$$

Note that $\hat{\mathfrak{D}}_0$ contains no non-zero $\hat{\mathfrak{D}}$-ideals: let X be a non-zero element of $\hat{\mathfrak{D}}$ and let $\mathbf{m} \in \mathbb{N}^n$ be such that $\mathbf{x}^{\mathbf{m}}$ occurs in some coefficient of X. Then it is easily seen that

$$\mathrm{ad}(\partial_1)^{m_1} \cdots \mathrm{ad}(\partial_n)^{m_n} X$$

is an element of $\hat{\mathfrak{D}}_{-1} \backslash \hat{\mathfrak{D}}_0$; here we use the fact that $\mathrm{char}\, K = 0$. We have now introduced all terminology needed for the notion of transitive Lie algebras.

DEFINITION 1.1.1. A subalgebra $\mathfrak{l}$ of $\hat{\mathfrak{D}}^{(n)}$ is called *transitive* if $\mathfrak{l} \cap \hat{\mathfrak{D}}_0$ has codimension n in $\mathfrak{l}$, and *intransitive* otherwise.

For any subalgebra $\mathfrak{l}$ of $\hat{\mathfrak{D}}^{(n)}$ we have $\mathrm{codim}_{\mathfrak{l}}(\mathfrak{l} \cap \hat{\mathfrak{D}}_0^{(n)}) \leq \mathrm{codim}_{\hat{\mathfrak{D}}^{(n)}} \hat{\mathfrak{D}}_0^{(n)} = n$; $\mathfrak{l}$ is transitive if and only if equality holds. A note regarding the terminology 'transitive' is in order here. We can think of a subalgebra $\mathfrak{l}$ of $\hat{\mathfrak{D}}^{(n)}$ as describing the infinitesimal action of a Lie group near the point 0. The condition that $\mathfrak{l}$ be transitive means that evaluation at 0 maps $\mathfrak{l}$ surjectively to the tangent space at 0, i.e., the Lie group action moves the origin in all directions.

This completes one point of view on formal transitive differential geometry. The second deals with pairs $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a Lie algebra over K and $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. Such pairs form a category, in which the morphisms $(\mathfrak{g}_1, \mathfrak{k}_1) \to (\mathfrak{g}_2, \mathfrak{k}_2)$ are the Lie algebra homomorphisms $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ with the property that $\phi^{-1}(\mathfrak{k}_2) = \mathfrak{k}_1$; such a ϕ induces an embedding $\mathfrak{g}_1/\mathfrak{k}_1 \to \mathfrak{g}_2/\mathfrak{k}_2$ of vector spaces. The following definition relates the pairs $(\mathfrak{g}, \mathfrak{k})$ to formal vector fields.

DEFINITION 1.1.2. A *realization of* $(\mathfrak{g}, \mathfrak{k})$ *in* n *variables* is a morphism $\phi : (\mathfrak{g}, \mathfrak{k}) \to (\hat{\mathfrak{D}}^{(n)}, \hat{\mathfrak{D}}_0^{(n)})$. The realization is called *transitive* if the image is transitive in the sense of Definition 1.1.1. The coefficients of derivations in the image $\phi(\mathfrak{g})$ are called the *coefficients of* ϕ.

A realization of $(\mathfrak{g}, \mathfrak{k})$ in n variables induces an embedding of vector spaces $\mathfrak{g}/\mathfrak{k} \to \hat{\mathfrak{D}}^{(n)}/\hat{\mathfrak{D}}_0^{(n)}$. As the latter vector space is n-dimensional, the pair $(\mathfrak{g}, \mathfrak{k})$ can only have a realization in n variables if $\mathrm{codim}_{\mathfrak{g}} \mathfrak{k} \leq n$. Moreover, a realization of $(\mathfrak{g}, \mathfrak{k})$ in n variables is transitive if and only if $\mathrm{codim}_{\mathfrak{g}} \mathfrak{k} = n$.

Now suppose that we have a transitive realization ϕ of $(\mathfrak{g}, \mathfrak{k})$ in n variables. Then $\ker \phi$ is the maximal $\mathfrak{g}$-ideal $\mathfrak{i}$ contained in $\mathfrak{k}$ (which exists because the sum of all $\mathfrak{g}$-ideals contained in $\mathfrak{k}$ is again a $\mathfrak{g}$-ideal, and contained in $\mathfrak{k}$). Indeed, we have

$$\ker \phi = \phi^{-1}(0) \subseteq \phi^{-1}(\hat{\mathfrak{D}}_0) = \mathfrak{k},$$

so that $\ker \phi \subseteq \mathfrak{i}$. Conversely, as $\phi(\mathfrak{g})$ is transitive, $\hat{\mathfrak{D}}$ is spanned by $\phi(\mathfrak{g})$ and $\hat{\mathfrak{D}}_0$, so that $U(\hat{\mathfrak{D}}) = U(\hat{\mathfrak{D}}_0)U(\phi(\mathfrak{g}))$, where $U(\mathfrak{l})$ denotes the universal enveloping algebra of $\mathfrak{l}$. Hence, with respect to the $U(\hat{\mathfrak{D}})$-module structure on $\hat{\mathfrak{D}}$ extending the adjoint action of $\hat{\mathfrak{D}}$ on itself we have

$$U(\hat{\mathfrak{D}})\phi(\mathfrak{i}) = U(\hat{\mathfrak{D}}_0)U(\phi(\mathfrak{g}))\phi(\mathfrak{i}) = U(\hat{\mathfrak{D}}_0)\phi(\mathfrak{i}) \subseteq \hat{\mathfrak{D}}_0,$$

where the second equality follows from the fact that $\mathfrak{i}$ is a $\mathfrak{g}$-ideal. We conclude that $U(\hat{\mathfrak{D}})\phi(\mathfrak{i})$ is a $\hat{\mathfrak{D}}$-ideal contained in $\hat{\mathfrak{D}}_0$, whence zero by an earlier observation.

We thus find that a transitive realization ϕ of a pair $(\mathfrak{g}, \mathfrak{k})$ is injective if and only if $\mathfrak{k}$ contains no non-zero $\mathfrak{g}$-ideal; compare this with the fact that the action of a group G on a homogeneous space G/H is faithful if and only if H contains no non-trivial normal subgroups of G. This analogy gives rise to the following definition.

DEFINITION 1.1.3. The pair $(\mathfrak{g}, \mathfrak{k})$ is called *effective* if $\mathfrak{k}$ contains no non-zero $\mathfrak{g}$-ideal. The pair $(\mathfrak{g}/\mathfrak{i}, \mathfrak{k}/\mathfrak{i})$, where $\mathfrak{i}$ is the maximal $\mathfrak{g}$-ideal contained in $\mathfrak{k}$, is called the *effective quotient* of $(\mathfrak{g}, \mathfrak{k})$. The pair $(\mathfrak{g}, \mathfrak{k})$ is called *primitive* if $\mathfrak{k}$ is maximal in $\mathfrak{g}$, and *imprimitive* otherwise. It is called *finite-dimensional* if $\mathfrak{g}$ is finite-dimensional. The codimension of $\mathfrak{k}$ in $\mathfrak{g}$ is called the *codimension of the pair* $(\mathfrak{g}, \mathfrak{k})$. A transitive subalgebra $\mathfrak{l}$ if $\hat{\mathfrak{D}}$ is called *primitive* if $(\mathfrak{l}, \mathfrak{l} \cap \hat{\mathfrak{D}}_0)$ is a primitive pair; otherwise, it is called *imprimitive.*

From the above it follows that transitive realizations of a pair $(\mathfrak{g}, \mathfrak{k})$ are in bijective correspondence with realizations of its effective quotient.

A few remarks on this definition are in order here. The first one concerns the word 'maximal': throughout this thesis, a subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is called maximal if $\mathfrak{k}$ is maximal, with respect to inclusion, among the proper subalgebras of $\mathfrak{g}$. Second, the maximal $\mathfrak{g}$-ideal $\mathfrak{i}$ contained in $\mathfrak{k}$ can be computed using the *Weisfeiler filtration* [**30**]: define $\mathfrak{g}_{-1} = \mathfrak{g}$, $\mathfrak{g}_0 := \mathfrak{k}$, and for $d \geq 0$,

$$\mathfrak{g}_{d+1} := \{X \in \mathfrak{g}_d \mid [X, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_d\};$$

then $\mathfrak{i} = \bigcap_{d=-1}^{\infty} \mathfrak{g}_d$. Moreover, the Weisfeiler filtration of $(\hat{\mathfrak{D}}, \hat{\mathfrak{D}}_0)$ coincides with the filtration introduced earlier, and a transitive realization of $(\mathfrak{g}, \mathfrak{k})$ is a homomorphism $\mathfrak{g} \to \hat{\mathfrak{D}}$ of filtered Lie algebras. Third, we will usually not be interested in the whole category of pairs $(\mathfrak{g}, \mathfrak{k})$, but in its full subcategories consisting of pairs of a fixed (and finite) codimension. Put differently, we will only compare pairs of the same codimension.

There exists a different (weaker) notion of primitivity in the literature, studied by Golubitsky in [**21**]. The following example explains the motivation for this notion.

EXAMPLE 1.1.4. Let G be the Lie group $\mathrm{SL}_2(\mathbb{R})$ and let H be the Cartan subgroup $\{\mathrm{diag}(t, t^{-1}) \mid t \in \mathbb{R}^*\}$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. The normalizer $\mathrm{norm}_G H$ of H in G is the union of H and the set

$$\left\{ \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \mid t \in \mathbb{R}^* \right\}.$$

The group $\mathrm{norm}_G H$ is a maximal proper subgroup of G, so that the action of G on $G/\mathrm{norm}_G H$ is primitive. On the other hand, the Lie algebra of $\mathrm{norm}_G H$ is $\mathfrak{h}$. This motivates a definition of primitivity in which the pair $(\mathfrak{g}, \mathfrak{h})$ is primitive, even though $\mathfrak{h}$ is not a maximal subalgebra of $\mathfrak{g}$.

In this thesis, however, we shall always use the notion of primitivity of Definition 1.1.3. For a concrete transitive subalgebra $\mathfrak{l}$ of $\hat{\mathfrak{D}}$, the inclusion $\mathfrak{l} \to \hat{\mathfrak{D}}$ is a realization of the pair $(\mathfrak{l}, \mathfrak{l} \cap \hat{\mathfrak{D}}_0)$. The following theorem, due to Guillemin and Sternberg [**31**], shows that, conversely, any abstract pair has a transitive realization which is unique up to the natural action of $\mathrm{Aut}(K[[\mathbf{x}]])$ on realizations: for $X \in \hat{\mathfrak{D}}$ and $\theta \in \mathrm{Aut}(K[[\mathbf{x}]])$, the map $\theta X \theta^{-1} \in \mathrm{End}_K(K[[\mathbf{x}]])$ is also an element of $\hat{\mathfrak{D}}$. Hence, if $\phi : \mathfrak{g} \to \hat{\mathfrak{D}}$ is a transitive realization of $(\mathfrak{g}, \mathfrak{k})$, then the map $X \mapsto \theta \phi(X) \theta^{-1}$ is also a transitive realization of $(\mathfrak{g}, \mathfrak{k})$.

THEOREM 1.1.5 (Realization Theorem). *Let $\mathfrak{g}$ be a Lie algebra over K and let $\mathfrak{k}$ be a subalgebra of finite codimension n. Then there exists a transitive realization $\phi : \mathfrak{g} \to \hat{\mathfrak{D}}^{(n)}$ of the pair $(\mathfrak{g}, \mathfrak{k})$. Moreover, if ψ is another such realization, then there exists a unique automorphism θ of $K[[\mathbf{x}]]$ such that $\theta \circ \phi(X) = \psi(X) \circ \theta$ for all $X \in \mathfrak{g}$.*

By this theorem, a proof of which is reviewed in Section 2.3, two effective pairs $(\mathfrak{g}_1, \mathfrak{k}_1)$ and $(\mathfrak{g}_2, \mathfrak{k}_2)$ are isomorphic if and only if their transitive realizations are related by a coordinate change. This allows us to go back and forth between the abstract point of view and the concrete one, and reason on whichever side is most convenient. For example, to prove that a transitive subalgebra $\mathfrak{l}$ of $\hat{\mathfrak{D}}^{(n)}$ is maximal among the finite-dimensional transitive Lie algebras in n variables is the same as proving that the transitive pair $(\mathfrak{l}, \mathfrak{l} \cap \hat{\mathfrak{D}})$ is maximal among the finite-dimensional pairs of codimension n, where, in the latter case, we call a pair $(\mathfrak{g}_1, \mathfrak{k}_1)$ a subpair of the pair $(\mathfrak{g}_2, \mathfrak{k}_2)$ if there exists an injective morphism from the former pair to the latter. This approach is followed in Chapter 3.

1.2. History of the Subject

Sophus Lie. At the end of the 19th century, Sophus Lie devised his theory of symmetries of differential equations [**42**]. For an informal discussion of his work, consider the following fourth order ordinary differential equation (o.d.e.):

$$y^{(4)} - \frac{5(y^{(3)})^2}{3y^{(2)}} - (y^{(2)})^{5/3} = 0,$$

where $y^{(i)}$ stands for the i-th derivative of the dependent variable y with respect to the independent variable x. It is clear from the equation that its set of integral curves in the (x, y)-plane is invariant under translation in the x-direction. The infinitesimal generator of this operation, the vector field ∂_x, is therefore called a *symmetry* of the equation. Similarly, the infinitesimal generator $x\partial_y - y\partial_x$ of rotation about the origin is a symmetry of the equation, as is made plausible by Figure 1 on the following page. This figure also shows that Lie symmetries describe *local* phenomena: there is no claim whatsoever that the set of maximal integral curves is invariant under the group of all rotations about the origin.

Lie showed that the symmetries of any o.d.e. of order at least 2 form a finite-dimensional Lie algebra of vector fields in the variables x and y, and obtained a system of linear partial differential equations, called the *determining system* [**46**], whose solution space is precisely the Lie algebra of symmetries of the o.d.e. In the o.d.e. at hand, the solution of this system is the Lie algebra

$$\langle \partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x \rangle.$$
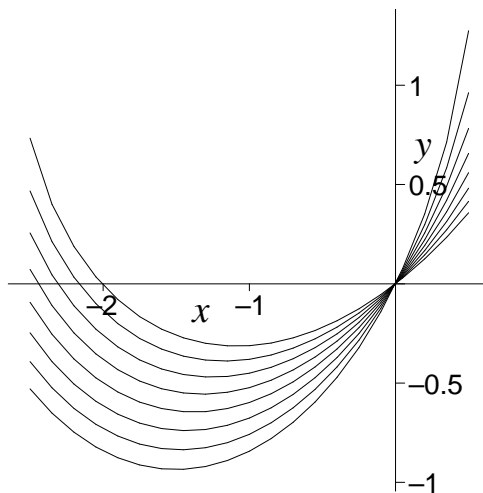
FIGURE 1. Some integral curves of the o.d.e.

This motivated Lie to pursue the classification of all finite-dimensional Lie algebras of vector fields in two variables. Slightly more accurately, his classification concerned Lie algebras $\mathfrak{l}$ of local vector fields near a fixed origin 0, which was assumed to be a 'point of regularity': for p in a neighbourhood of 0, the dimension of $\mathfrak{l}|_p$ should be constant. First, Lie classified the Lie algebras of vector fields in one variable, and found that there are three classes of them, namely $\langle \partial_x \rangle$, $\langle \partial_x, x\partial_x \rangle$ and $\langle \partial_x, x\partial_x, x^2\partial_x \rangle$. For the case of two variables, Lie distinguished between transitive Lie algebras, for which the dimension of $\mathfrak{l}|_0$ equals 2, or equivalently: the codimension of the isotropy subalgebra $\mathfrak{l}_0$ in $\mathfrak{l}$ is 2; and the intransitive ones, for which this is not the case. He put most effort into classifying Lie algebras of the former type. Among these, he distinguished further between the imprimitive Lie algebras, for which the corresponding local Lie group leaves a foliation invariant, or equivalently: for which $\mathfrak{l}_0$ is not a maximal subalgebra of $\mathfrak{l}$; and the primitive ones, for which it is. Lie showed that there are only three primitive Lie algebras in two variables, and proceeded with the classification of the imprimitive ones by choosing an intermediate subalgebra $\mathfrak{p}$ between $\mathfrak{l}_0$ and $\mathfrak{l}$. The effective quotients of $(\mathfrak{l}, \mathfrak{p})$ and $(\mathfrak{p}, \mathfrak{l}_0)$ are then both among the three transitive Lie algebras in one variable. This leads to nine cases, which Lie handled one by one. A version of his classification [**41**] can be found in Tables 1 and 2 of Appendix A of this thesis.

Lie did not make entirely clear what he was classifying: algebras of smooth, analytic, real or complex vector fields, but his classification coincides with the classification of finite-dimensional formal transitive Lie algebras in two variables over an algebraically closed field of characteristic zero. Theorem 2.3.4 of this thesis explains why the latter classification is identical to that of finite-dimensional *convergent* transitive Lie algebras. Over the real numbers, the classification is slightly different [**23**]. It is worth noticing that by the Realization Theorem, the orbit of the Lie algebra $\mathfrak{l}$ of symmetries of an o.d.e. under coordinate changes is determined by the abstract pair $(\mathfrak{l}, \mathfrak{l}_0)$, provided that $\mathfrak{l}$ is transitive. This fact is exploited in [**17**] to determine the symmetry type of an o.d.e.

After classifying the finite-dimensional transitive Lie algebras in two variables, Lie proceeded with the case of three variables in much the same way; let us formulate this approach for formal transitive Lie algebras in three variables. By the Realization Theorem, these are parameterized by pairs $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ of codimension 3 that contains no $\mathfrak{g}$-ideal. We distinguish three (possibly non-disjoint) cases: either $\mathfrak{k}$ is maximal in $\mathfrak{g}$, or there exists a maximal subalgebra $\mathfrak{k}_1$ of $\mathfrak{g}$ that contains $\mathfrak{k}$ as a maximal subalgebra, or there exist subalgebras $\mathfrak{k}_1, \mathfrak{k}_2$ of $\mathfrak{g}$ such that

$$\mathfrak{g} \supsetneq \mathfrak{k}_2 \supsetneq \mathfrak{k}_1 \supsetneq \mathfrak{k}.$$

First, Lie handled the primitive Lie algebras in three variables. Let us estimate the amount of work needed to complete the classification from this point. In the second case, there are two possibilities: $\operatorname{codim}_{\mathfrak{g}} \mathfrak{k}_1 = 1$ or 2. In either case, the effective quotients of $(\mathfrak{g}, \mathfrak{k}_1)$ and $(\mathfrak{k}_1, \mathfrak{k})$ are primitive, and there are three possibilities for each. This leads to $2 * 3 * 3 = 18$ cases to be considered one by one, and Sophus Lie did so. In the third case, there are three possibilities for each of the effective quotients of $(\mathfrak{g}, \mathfrak{k}_2), (\mathfrak{k}_2, \mathfrak{k}_1)$ and $(\mathfrak{k}_1, \mathfrak{k})$; this leads to 27 cases. Lie claims to have performed this task as well, but found the result too lengthy for publication. He did, however, notice the remarkable phenomenon that each class of transitive Lie algebras in one, two or three variables has a representative in which each coefficient is a polynomial in the x_i and some exponentials $\exp(\lambda x_i)$. This fact, and its conjectural generalization to higher dimensions, is treated in Chapter 2.

Morozov and Dynkin. While classification of finite-dimensional transitive Lie algebras in n variables seems, even for $n = 4$ or $n = 5$, a hopeless job, the more modest task of classifying the primitive ones among them is feasible. Indeed, over an algebraically closed field of characteristic 0, this classification was completed by Morozov and Dynkin in the late 1930s and the early 1950s, respectively [18], [19], [45]. In the remainder of this section, we assume that K indeed is algebraically closed, and of characteristic 0 as always.

By a relatively easy argument, which can also be found in [21], Morozov showed that if $(\mathfrak{g}, \mathfrak{k})$ is a primitive effective pair and $\mathfrak{g}$ is not simple, then either $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an Abelian ideal on which $\mathfrak{k}$ acts faithfully and irreducibly—so that $\mathfrak{k}$ is semisimple plus, possibly, a one-dimensional center—or $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are isomorphic simple Lie algebras, and $\mathfrak{k}$ is the diagonal subalgebra.

If $\mathfrak{g}$ is simple, then by a result of Karpelevich a maximal subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ either is reductive or contains a Borel subalgebra of $\mathfrak{g}$ [38]; in the latter case $\mathfrak{k}$ is called a *parabolic* subalgebra of $\mathfrak{g}$. The classification of the reductive maximal subalgebras of simple Lie algebras was completed by Dynkin. A few lines in this introduction would not do justice to his results, so I refer to Chapter 3 for a detailed account of them.

This leaves the parabolic subalgebras of simple Lie algebras. As we shall encounter these in several chapters, let us describe them in detail. The most convenient way to do so is in terms of the root system, and this description applies to general semisimple Lie algebras. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\Delta \subseteq \mathfrak{h}^*$ the root system. For $\alpha \in \Delta$, denote by $\mathfrak{g}_\alpha$ the corresponding root space. Choose a fundamental system $\Pi \subseteq \Delta$ and denote by $\Delta_\pm$ the corresponding sets of positive and negative roots. Let Π_0 be a subset of Π and let Δ_0 be the intersection of the $\mathbb{Z}$-linear span of Π_0 and Δ.

Define

$$\mathfrak{p}_{\Pi_0} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0 \cup \Delta_+} \mathfrak{g}_\alpha;$$

clearly $\mathfrak{p}_{\Pi_0}$ contains the Borel subalgebra

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

Conversely, any parabolic subalgebra of $\mathfrak{g}$ is conjugate, by an inner automorphism, to a unique parabolic subalgebra of the form $\mathfrak{p}_{\Pi_0}$ ([**34**], page 88). The maximal parabolic subalgebras of $\mathfrak{g}$ are obtained by taking $\Pi_0 = \Pi \setminus \{\beta\}$ for some $\beta \in \Pi$; in this case we shall write $\mathfrak{p}_\beta$ for $\mathfrak{p}_{\Pi_0}$, or $\mathfrak{p}_i$, where i is the label of the node corresponding to β in the Dynkin diagram, in the standard labelling of [**7**]. Note that $\mathfrak{p}_\beta \neq \mathfrak{p}_{\{\beta\}}$; this disadvantage of our notation is compensated by the frequent appearance of maximal parabolic subalgebras in Chapter 3.

Guillemin, Sternberg, and Blattner. As mathematics evolves, it gets harder and harder to give credit to everyone who contributed to the creation of new theory. This is reflected by the increasing number of mathematicians in the titles of subsections in the current section. I have chosen, however, to pay most attention to the work of mathematicians that directly inspired my research for this thesis.

The infinite-dimensional counterpart of the classification of primitive Lie algebras over an algebraically closed field was initiated by Cartan, and his results were proved in more sophisticated ways by Weisfeiler in [**61**] and by Guillemin in [**30**]. Together with Sternberg, Guillemin had earlier described a general framework for formal transitive differential geometry [**31**]; in particular, the Realization Theorem is proved in that paper. A few years later, Blattner gave a constructive proof of that theorem in [**4**].

This thesis does not treat infinite-dimensional transitive Lie algebras, but it does use Guillemin and Sternberg's framework for finite-dimensional ones. Indeed, Section 1.1 is based entirely on their ideas.

1.3. Topics of this Thesis

Realizations with Nice Coefficients. Once we are able to compute realizations of abstract pairs $(\mathfrak{g}, \mathfrak{k})$ in terms of vector fields with formal power series coefficients, we wonder whether realizations with coefficients in certain subalgebras $A \subseteq K[[\mathbf{x}]]$ exist. We shall encounter the following subalgebras of $K[[\mathbf{x}]]$.

(1) $K[\mathbf{x}]$, the algebra of polynomials in the x_i. A realization with coefficients in $K[\mathbf{x}]$ is called *polynomial*. Its image can be seen as a subalgebra of the Lie algebra $\mathfrak{D}^{(n)} = \mathfrak{D}$ of polynomial vector fields.

(2) Algebraic functions: suppose that K is algebraically closed and that p is a simple point on an algebraic variety V over K of dimension n. Then the completion of the local ring $\mathcal{O}_p$, i.e., the stalk at p of the sheaf of regular functions on V, is isomorphic to $K[[\mathbf{x}]]$ ([**33**], page 34), so we may view the elements of $\mathcal{O}_p$ as formal power series satisfying some algebraic relation with coefficients in $K[\mathbf{x}]$.

(3) Convergent power series with respect to an Archimedean valuation on K.

(4) Somewhat less common: the algebra $E^{(n)}$, defined by

$$E^{(n)} := \{f \in K[[\mathbf{x}]] \mid \forall i \in \{1, \ldots, n\} \exists P \in K[t] : P(\partial_i)f = 0\}.$$

If n is irrelevant or clear from the context, we write E for $E^{(n)}$. In words, an element of E satisfies a linear ordinary differential equation with constant coefficients in each of its variables. If K is algebraically closed, E consists precisely of all polynomials in the x_i and the $\exp(\lambda x_i)$ for $\lambda \in K$. The algebra E plays a role in a conjecture of Lie connected with his observation on the transitive Lie algebras in three variables that he did not publish; see Chapter 2.

A guiding question in this thesis is: given a pair $(\mathfrak{g}, \mathfrak{k})$, what type of coefficients do we need for a realization? Blattner's proof of the Realization Theorem gives rise to a fairly explicit Realization Formula, which is treated in Chapter 2. This formula defines a convergent realization with respect to any Archimedean valuation on K. Moreover, under certain conditions it yields polynomial coefficients or coefficients in E.

Inclusions among Primitive Lie Algebras. The work of Dynkin and Morozov settles the classification problem for finite-dimensional primitive Lie algebras. It is natural to ask which inclusions exist among them. More precisely, fix a natural number n, and consider all effective primitive pairs of codimension n; which injective morphisms exist among these? Let us consider an example.

EXAMPLE 1.3.1. Let V be a finite-dimensional vector space over K, and let $\mathfrak{d} = \{(X, X) \mid X \in \mathfrak{sl}(V)\}$ be the diagonal subalgebra of the Lie algebra $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$. Then the pair $(\mathfrak{sl}(V) \oplus \mathfrak{sl}(V), \mathfrak{d})$ is a subpair of the pair $(\mathfrak{sl}(V \otimes V^*), \mathfrak{p}_1)$. Indeed, $V \otimes V^*$ has the structure of an $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$-module defined by

$$(X, Y)(v \otimes f) := Xv \otimes f + v \otimes Yf$$

for all $X, Y \in \mathfrak{sl}(V)$, $v \in V$, and $f \in V^*$. This defines the embedding $\phi : \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \to \mathfrak{sl}(V \otimes V^*)$. On the other hand, we may identify $V \otimes V^*$ with $\mathfrak{gl}(V)$. Under this identification, we find

$$\begin{aligned}
(X, X)(v_0 \otimes f_0)(v) &= (Xv_0 \otimes f_0 + v_0 \otimes Xf_0)(v) \\
&= f_0(v)Xv_0 - f_0(Xv)v_0 \\
&= [X, v_0 \otimes f_0]v.
\end{aligned}$$

Hence, the action of $\mathfrak{d}$ on $V \otimes V^*$ is simply the adjoint action of $\mathfrak{sl}(V)$ on $\mathfrak{gl}(V)$. In this representation, $\mathfrak{sl}(V)$ leaves invariant the line spanned by the identity. We conclude that $\phi(\mathfrak{d})$ is contained in the stabilizer $\mathfrak{p}_1$ of a one-dimensional subspace of $V \otimes V^*$. As $\mathfrak{d}$ is a maximal proper subalgebra of $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$, this implies that $\phi^{-1}(\mathfrak{p}_1) = \mathfrak{d}$, i.e., ϕ is an injective morphism of pairs. Finally, the codimensions of both pairs at hand is $\dim(V)^2 - 1$.

A motivation for our quest for inclusions among pairs is the following: if $(\mathfrak{g}_1, \mathfrak{k}_1)$ is a subpair of the pair $(\mathfrak{g}_2, \mathfrak{k}_2)$ of the same codimension n, and if we have a transitive realization for the latter with nice coefficients (e.g., polynomials), then this realization restricts to a nice transitive realization of the former pair. In the above example, we thus find that the pair $(\mathfrak{sl} \oplus \mathfrak{sl}(V), \mathfrak{d})$ has a polynomial realization, because $(\mathfrak{sl}(V \otimes V^*), \mathfrak{p}_1)$ has one; see Example 2.4.1. Chapter 3 of this thesis presents Morozov's and Dynkin's results in more detail, describes some inclusions among primitive pairs, and proves some non-existence results on such inclusions.

Integration to Algebraic Group Actions. Suppose that K is algebraically closed. If $\mathfrak{g}$ is the Lie algebra of an affine algebraic group G over K and $\mathfrak{h}$ is the Lie algebra of a closed subgroup H of G, then we can find a realization of $(\mathfrak{g}, \mathfrak{h})$ as follows: differentiate the action of G on the homogeneous space G/H to an action of $\mathfrak{g}$ by derivations on the sheaf of regular functions of G/H. In particular, for an open affine neighbourhood U of eH in G/H we obtain a natural homomorphism from $\mathfrak{g}$ into the Lie algebra of derivations on the affine algebra $K[U]$, and the pre-image of the isotropy subalgebra at eH under this morphism is precisely $\mathfrak{h}$. This is the close relation between transitive group actions and transitive differential geometry alluded to in Section 1.1. The coefficients of this realization clearly depend on the geometry of U. For example, if U is isomorphic to an affine space, then we thus obtain a polynomial realization for the pair $(\mathfrak{g}, \mathfrak{h})$. The advantage of this approach over the Realization Formula based on Blattner's proof is that we can use the geometry of homogeneous spaces for our realization problem.

EXAMPLE 1.3.2. Let G be a reductive algebraic group and let P be a parabolic subgroup of G. Denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{p}$, respectively. By the Bruhat decomposition, the point eP on G/P has an open neighbourhood isomorphic to an affine space. Hence, by the above construction, the pair $(\mathfrak{g}, \mathfrak{p})$ has a polynomial realization.

On the other hand, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and we may choose $\mathfrak{h}, \Pi$ and Π_0 such that $\mathfrak{p} = \mathfrak{p}_{\Pi_0}$ in the notation of page 7. Then the subalgebra $\mathfrak{q}$ spanned by the $\mathfrak{g}_\alpha$ with $\alpha \in \Delta \setminus (\Delta_0 \cup \Delta_+)$ acts nilpotently on $\mathfrak{g}$, and is a vector space complement to $\mathfrak{p}$. Under these conditions we may use the Realization Formula of Chapter 2 to compute a polynomial realization of $(\mathfrak{g}, \mathfrak{p})$ explicitly; see Theorem 2.2.3.

In the setting of this example, we are lucky to find that the results obtained by the group action approach have a constructive counterpart. In general the situation is much more complex, and it is unclear how to use the Realization Formula to compute a polynomial realization whose existence is guaranteed by the group action approach.

The construction of realizations through algebraic group actions is a topic of chapter 4. However, the major topic of that chapter is the inverse question: suppose that we have a homomorphism from a Lie algebra $\mathfrak{g}$ into the Lie algebra of derivations on an affine algebraic variety U; does it correspond to an algebraic group action? The following example shows that we may have to let the group act on a variety properly containing U as an open dense subset.

EXAMPLE 1.3.3. Consider the embedding ϕ of $\mathfrak{sl}_2$ into $\mathrm{Der}_K(K[x])$ determined by

$$\phi(E) = -\partial_x, \ \phi(H) = -2x\partial_x, \text{ and } \phi(F) = x^2\partial_x,$$

where $\{E, H, F\}$ is the usual Chevalley basis of $\mathfrak{sl}_2$. The action of $\mathfrak{sl}_2$ on $K[x]$ via ϕ is not locally finite (see page 57 for a definition), as $\phi(F)$ increases the degree of polynomials. This shows that the above action of $\mathfrak{sl}_2$ does not come from an action of SL_2 on the affine line; see Section 4.4. However, ϕ satisfies

$$\exp(t\phi(E))(x) = x - t,$$
$$\exp(t\phi(H))(x) = \exp(-2t)x, \text{ and}$$
$$\exp(t\phi(F))(x) = \frac{x}{1 - tx},$$

where t is a variable and $\exp(tX)(f) \in K[x][[t]]$ is the formal power series (with coefficients in $K[x]$)

$$\sum_{i=0}^{\infty} X^i(f)\frac{t^i}{i!}$$

for any $f \in K[x]$ and $X \in \mathrm{Der}_K(K[x])$; in particular, all expressions above are rational functions in x, t, and certain exponentials. This observation allow us to apply the results of Section 4.5, and to deduce that this action of $\mathfrak{sl}_2$ can be integrated to an action of the group SL_2 on an algebraic variety containing the affine line as a open dense subset. Indeed, we use Weil's theory of pre-transformation spaces to recover, from the vector fields above, the projective line and the action of SL_2 on it by Möbius transformations.

Infinite-dimensional Representation Theory. Primitive pairs $(\mathfrak{g}, \mathfrak{p})$ where $\mathfrak{g}$ is simple and $\mathfrak{p}$ is parabolic are usually maximal among the pairs of their codimension. This is shown in Chapter 3, and was already proved by Onishchik in the 1960s [**48**]. In particular, if ϕ is the polynomial transitive realization of $(\mathfrak{g}, \mathfrak{p})$ into the Lie algebra $\mathfrak{D}$ of polynomial vector fields in $n := \mathrm{codim}_{\mathfrak{g}} \mathfrak{p}$ variables constructed in Example 1.3.2, then $\phi(\mathfrak{g})$ is usually maximal among the finite-dimensional subalgebras of $\mathfrak{D}$. This raises the following more subtle question: what is the structure of $\mathfrak{D}$ when viewed as a $\mathfrak{g}$-module through ϕ? The following example shows that this question involves the theory of Verma modules and the category $\mathcal{O}$ (see Chapter 5 for a definition).

EXAMPLE 1.3.4. Recall the realization $\phi : \mathfrak{sl}_2 \to \mathfrak{D}^{(1)}$ of Example 1.3.3, and view $\mathfrak{D}$ as an $\mathfrak{sl}_2$-module through ϕ. Clearly, $\phi(\mathfrak{sl}_2)$ is a submodule of $\mathfrak{D}$, and it is not hard to see that $\mathfrak{D}/\phi(\mathfrak{sl}_2)$ is an irreducible module generated by the element $x^3\partial_x + \phi(\mathfrak{sl}_2)$, which is a zero vector of E and has H-eigenvalue -4.

Chapter 5 presents a detailed description of the structure of $\mathfrak{D}$ as a $\mathfrak{g}$-module in the special case where $\mathfrak{p} = \mathfrak{p}_\beta$ for a simple root β that has coefficient 1 in the highest root of $\mathfrak{g}$. We show that in this case the $\mathfrak{g}$-module $\mathfrak{D}$ has a finite *composition chain*, i.e., a finite chain

$$\mathfrak{D} = M_1 \supsetneq M_2 \supsetneq \ldots \supsetneq M_l = 0$$

of $\mathfrak{g}$-submodules such that the quotient $L_i := M_i/M_{i+1}$ of M is irreducible for all $i = 1, \ldots, l-1$. Furthermore, we derive a formula for the multiplicity of a given irreducible module L among the L_i, and formulate a conjecture regarding these multiplicities.

CHAPTER 2

Blattner's Construction and a Conjecture of Lie

2.1. Introduction

Sophus Lie conjectured that any complex finite-dimensional transitive Lie algebra of vector fields in the variables $x_1, \ldots, x_n$ has a conjugate whose coefficients lie in the algebra E generated by the x_i and the exponentials $\exp(\lambda x_i)$ for $\lambda \in \mathbb{C}$. This chapter treats this conjecture in the setting of formal power series. First, we derive a formula that realizes an abstractly given transitive Lie algebra in terms of vector fields. We establish sufficient conditions for this formula to yield only polynomial coefficients; they slightly generalize a known result. Next, we present a sufficient, but strong, condition for the output to contain only coefficients in E. Finally, we prove Lie's conjecture for $n = 1, 2$, and 3. In the first two cases, this result is not new, as Lie completely classified the transitive Lie algebras in those dimensions. For $n = 3$, the result can be considered new, as Lie did not publish his complete classification in three dimensions. The principal merit of our approach, however, is that we need not classify the transitive Lie algebras in 3 variables. In particular, the sub-case for $n = 3$ that did not appear in print because Lie claimed that it was too lengthy, happens to satisfy our strong condition and can therefore be handled without any further effort.

Nevertheless, our method of proving Lie's conjecture for small n comprises many *ad hoc* arguments that do not seem to generalize to larger n. Although it is dangerous to base one's intuition upon this mere fact, I do not believe Lie's conjecture in full generality. In order to illustrate why, consider the analogous question with E replaced by the polynomial ring $K[x_1, \ldots, x_n]$. For $n = 1$, it is well known that any finite-dimensional transitive Lie algebra has a conjugate with only polynomial coefficients, but in Section 2.7 we show that some of the algebras from Lie's list for $n = 2$ do not have such a conjugate. To this end, we employ the transcendence degree to distinguish between conjugates of the polynomial ring inside $K[[x_1, \ldots, x_n]]$ and 'wilder' algebras of formal power series. Unfortunately, we do not yet have a similar tool to distinguish conjugates of E from even wilder algebras.

This chapter is an expanded version of [16]; I thank Marius van der Put for numerous useful comments on an earlier version of that paper, and Heike Gramberg for her help in proving Theorem 2.3.4.

2.2. Lie's Conjecture

> It turns out, that every transitive group of 3-space with coordinates x, y, z can be brought to a form in which the coefficients of p, q, and r (Lie's notation for ∂_x, ∂_y, and ∂_z—J.D.) are entire functions of x, y, z, and particular exponential expressions $e^{\lambda_1}, e^{\lambda_2}, \ldots$, where $\lambda_1, \lambda_2, \ldots$ denote linear functions of x, y, z. Very probably, a similar statement holds for the transitive groups of n-space.
>
> Translated from: Sophus Lie, [**42**], page 177.

In the formal power series setting, Lie's conjecture can be formulated as follows. Let K be a field of characteristic zero; this will be the ground field of all Lie algebras and vector spaces in this chapter, unless explicitly stated otherwise. Let n be a positive integer, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a list of indeterminates, and $\hat{\mathfrak{D}}^{(n)}$ the Lie algebra of derivations of $K[[\mathbf{x}]]$. The automorphism group $\operatorname{Aut} K[[\mathbf{x}]]$ acts on $\hat{\mathfrak{D}}^{(n)}$ by conjugation, as we have seen in Chapter 1. If $K = \mathbb{C}$, then the algebra $E^{(n)}$ of page 8 is precisely the algebra of functions that Lie refers to in his conjecture. Hence, the following is a natural formulation of Lie's conjecture in the setting of formal power series with coefficients from an arbitrary field of characteristic 0.

CONJECTURE 2.2.1. *For any finite-dimensional transitive subalgebra $\mathfrak{g}$ of $\hat{\mathfrak{D}}^{(n)}$, there exists an automorphism ψ of $K[[\mathbf{x}]]$ such that $\psi \mathfrak{g} \psi^{-1}$ is a subalgebra of*

$$E^{(n)}\partial_1 \oplus \ldots \oplus E^{(n)}\partial_n.$$

By the Realization Theorem, this conjecture is equivalent to the following one.

CONJECTURE 2.2.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ of codimension n. Then the pair $(\mathfrak{g}, \mathfrak{k})$ has a transitive realization with coefficients in $E^{(n)}$.*

We shall prove the following theorems in favour of Lie's conjecture.

THEOREM 2.2.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\mathfrak{k}$ and $\mathfrak{m}$ be subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as vector spaces. Assume that $\mathfrak{m}$ acts nilpotently on $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ has a transitive realization with polynomial coefficients.*

THEOREM 2.2.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and suppose that it has a sequence*

$$\mathfrak{g} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \ldots \supset \mathfrak{g}_0 = \mathfrak{k}$$

of subalgebras, with $\dim \mathfrak{g}_i = \dim \mathfrak{k} + i$. Then $(\mathfrak{g}, \mathfrak{k})$ has a transitive realization with coefficients in $E^{(n)}$.

THEOREM 2.2.5. *Suppose that K is algebraically closed. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ of codimension $n \in \{1, 2, 3\}$. Then $(\mathfrak{g}, \mathfrak{k})$ has a transitive realization with coefficients in $E^{(n)}$.*

The remainder of this chapter is organized as follows. In Section 2.3, we derive an explicit transitive realization $\phi_{\mathbf{Y}}$, which depends on the choice of an ordered basis $\mathbf{Y} = (Y_1, \ldots Y_n)$ complementary to $\mathfrak{k}$. If K is endowed with a valuation, then the coefficients of $\phi_{\mathbf{Y}}$ turn out to be convergent power series; see Proposition 2.3.3. Moreover, two

transitive Lie algebras with convergent coefficients are conjugate under a convergent coordinate change if and only if they are conjugate under a formal coordinate change; see Theorem 2.3.4.

In Section 2.4, we use $\phi_{\mathbf{Y}}$ to prove Theorem 2.2.3, and in Section 2.5 we use it to prove Theorem 2.2.4. Section 2.6 applies the techniques from the previous two sections to prove Theorem 2.2.5. Finally, Section 2.7 briefly discusses Lie's conjecture in more variables. The text is larded with GAP-sessions in which explicit realizations are computed.

2.3. A Realization Formula

In [**4**], Blattner proves the Realization Theorem in a very constructive way. In this section we make his realization even more explicit. To this end, let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ of codimension n. Choose a basis $X_1, \ldots, X_k, Y_1, \ldots, Y_n$ of $\mathfrak{g}$, such that the X_i span $\mathfrak{k}$. By the Poincaré-Birkhoff-Witt theorem ([**35**], Chapter V), the monomials

$$\mathbf{X}^{\mathbf{r}}\mathbf{Y}^{\mathbf{s}} := X_1^{r_1} \cdot \ldots \cdot X_k^{r_k} \cdot Y_1^{s_1} \cdot \ldots \cdot Y_n^{s_n}$$

are a basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$; these monomials are called *PBW-monomials*. For $u \in U(\mathfrak{g})$ and $i \in \{1, \ldots, n\}$, let $\chi_i(u)$ be the coefficient of the PBW-monomial Y_i in u, when the latter is written as a linear combination of this basis.

THEOREM 2.3.1 (Realization Formula). *The map $\phi_{\mathbf{Y}} : \mathfrak{g} \to \hat{\mathfrak{D}}^{(n)}$ defined by*

$$(1) \qquad \phi_{\mathbf{Y}}(X) := \sum_{i=1}^{n} \left(\sum_{\mathbf{m} \in \mathbb{N}^n} \chi_i(\mathbf{Y}^{\mathbf{m}}X) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} \right) \partial_i, \quad X \in \mathfrak{g},$$

where $\mathbf{m}! := \prod_{i=1}^{n} m_i!$ and $\mathbf{Y}^{\mathbf{m}}X$ is the element of $U(\mathfrak{g})$ obtained by multiplying $\mathbf{Y}^{\mathbf{m}}$ from the right with X, is a transitive realization of the pair $(\mathfrak{g}, \mathfrak{k})$.

PROOF. We give an outline of Blattner's construction, and prove that it leads to (1). Define the $\mathfrak{g}$-module

$$A := \operatorname{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), K)$$

as follows: the universal enveloping algebra $U(\mathfrak{g})$ is a left $U(\mathfrak{k})$-module by multiplication from the left, K is viewed as a trivial left $U(\mathfrak{k})$-module, and A is the space of $U(\mathfrak{k})$-module homomorphisms from the former to the latter module. An element $X \in \mathfrak{g}$ acts on such a homomorphism $\phi \in A$ by

$$(X\phi)(u) := \phi(uX) \text{ for } u \in U(\mathfrak{g}),$$

and it is straightforward to check that A does indeed become a $\mathfrak{g}$-module in this way. Moreover, Blattner defines a commutative K-bilinear multiplication on A such that $\mathfrak{g}$ acts on A by K-linear derivations.

Let $\alpha : K[x_1, \ldots, x_n] \to U(\mathfrak{g})$ be the linear monomorphism determined by

$$\alpha(\mathbf{x}^{\mathbf{m}}) = \mathbf{Y}^{\mathbf{m}}.$$

From the PBW-theorem it follows that $U(\mathfrak{g})$ is a free $U(\mathfrak{k})$-module of which the monomials $\mathbf{Y}^{\mathbf{m}}$ form a basis. Hence, any $U(\mathfrak{k})$-homomorphism from $U(\mathfrak{g})$ to K is determined by its values on the PBW-monomials $\mathbf{Y}^{\mathbf{m}}$ (in fact, such a homomorphism vanishes on all

PBW-monomials containing at least one of the X_i), and those values can be prescribed arbitrarily. It follows that the pullback

$$\alpha^* : A \to \mathrm{Hom}_K(K[x], K) = K[x]^*$$

is a linear isomorphism. Finally, the spaces $K[x]^*$ and $K[[x]]$ are identified by

$$\beta : \sum_{\mathbf{m}} c_{\mathbf{m}} f_{\mathbf{m}} \mapsto \sum_{\mathbf{m}} \frac{c_{\mathbf{m}}}{\mathbf{m}!} \mathbf{x}^{\mathbf{m}},$$

where the $f_{\mathbf{m}} \in K[x]^*$ are determined by $f_{\mathbf{m}}(x^{\mathbf{r}}) = \delta_{\mathbf{m},\mathbf{r}}$.

We have thus constructed a linear isomorphism $\beta \circ \alpha^* : A \to K[[\mathbf{x}]]$, and one can show that it is an isomorphism of algebras. Hence, $\mathfrak{g}$ acts on $K[[\mathbf{x}]]$ by derivations; let $\phi : \mathfrak{g} \to \hat{\mathfrak{D}}^{(n)}$ denote this representation. To make the action explicit, let $X \in \mathfrak{g}$, $i \in \{1, \ldots, n\}$, and compute $\phi(X)x_i$ as follows. We have $\beta^{-1} x_i = f_{\mathbf{e}_i}$, where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{N}^n$. Next, $(\alpha^*)^{-1} f_{\mathbf{e}_i}$ is the $U(\mathfrak{k})$-homomorphism mapping Y_i to 1 and all other monomials $\mathbf{Y}^{\mathbf{m}}$ to zero, hence $(\alpha^*)^{-1} f_{\mathbf{e}_i} = \chi_i$. By definition of the $\mathfrak{g}$-action on A, one has

$$(X\chi_i)(\mathbf{Y}^{\mathbf{m}}) = \chi_i(\mathbf{Y}^{\mathbf{m}} X).$$

Hence, $\alpha^*(X\chi_i)$ maps $\mathbf{x}^{\mathbf{m}}$ to $\chi_i(\mathbf{Y}^{\mathbf{m}} X)$. It follows that

$$\phi(X)x_i = \beta(\alpha^*(X\chi_i)) = \sum_{\mathbf{m}} \chi_i(\mathbf{Y}^{\mathbf{m}} X) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}.$$

By continuity of derivations on $K[[\mathbf{x}]]$, this suffices to conclude that

$$\phi(X) = \sum_i \left(\sum_{\mathbf{m}} \chi_i(\mathbf{Y}^{\mathbf{m}} X) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} \right) \partial_i,$$

as claimed in the theorem.

Finally, note that $\chi_i(\mathbf{Y}^0 Y_j) = \delta_{ij}$ and that $\chi_i(\mathbf{Y}^0 X) = 0$ for all $i, j = 1, \ldots, n$ and $X \in \mathfrak{k}$. This proves that $\phi^{-1}(\hat{\mathfrak{D}}_0^{(n)}) = \mathfrak{k}$, so that ϕ is a transitive realization of $(\mathfrak{g}, \mathfrak{k})$, as claimed.   □

The Realization Formula depends heavily on $\mathbf{Y} = (Y_1, \ldots, Y_n)$, but not on the choice of basis for $\mathfrak{k}$. Indeed, one can write elements of $U(\mathfrak{g})$ in a unique way as $\sum_{\mathbf{m}} u_{\mathbf{m}} \mathbf{Y}^{\mathbf{m}}$ with $u_{\mathbf{m}} \in U(\mathfrak{k})$, and each $u_{\mathbf{m}}$ can be uniquely written as $c_{\mathbf{m}} \cdot 1 + u'_{\mathbf{m}}$, where $c_{\mathbf{m}} \in K$ and $u'_{\mathbf{m}} \in \mathfrak{k}U(\mathfrak{k})$. The Realization Formula needs only the $c_{\mathbf{m}}$ for particular $\mathbf{m}$; they are independent of a choice of basis for $\mathfrak{k}$. This justifies the notation $\phi_{\mathbf{Y}}$.

EXAMPLE 2.3.2. Let $\mathfrak{g}$ be $\mathfrak{sl}_2$ with Chevalley basis F, H, E and relations $[H, F] = -2F, [E, F] = H$, and $[H, E] = 2E$. Let $\mathfrak{k}$ be the Borel subalgebra spanned by H, F. Define $X_1 = F, X_2 = H, Y_1 = E$. Then

$$\chi_1(Y_1^m Y_1) = \delta_{0,m},$$

and, for $m \geq 1$,

$$Y_1^m X_2 = E^m H = E^{m-1} H E - 2E^m.$$

The former term does not contribute to the coefficient of Y_1, so that (also for $m = 0$)

$$\chi_1(Y_1^m X_2) = -2\delta_{1,m}.$$

Finally, for $m \geq 2$,
$$Y_1^m X_1 = E^m F = E^{m-1} F E + E^{m-1} H;$$
hence (also for $m = 0, 1$):
$$\chi_1(Y_1^m X_1) = -2\delta_{2,m}.$$
After division by 2! in the last case, we find the realization
$$F \mapsto -x_1^2 \partial_1, H \mapsto -2x_1 \partial_1, E \mapsto \partial_1.$$

I implemented the Realization Formula in the computer algebra package GAP [**20**], using De Graaf's algorithms for Lie algebras [**24**],[**25**]. For comparison, we include a GAP-session dealing with this example.

```
gap> g:=SimpleLieAlgebra("A",1,Rationals);;
gap> B:=BasisByGenerators(g,GeneratorsOfAlgebra(g){[2,3,1]});;
gap> Blattner(g,B,1,3);
[ [ [(-1)*x_1^2*D_1], [(-2)*x_1*D_1], [(1)*D_1] ],
  <algebra-with-one of dimension infinity over Rationals> ]
```

First, $\mathfrak{g}$ is assigned to the variable g. This predefined Lie algebra has (E, F, H) as default basis; the permuted basis $(X_1, X_2, Y_1) = (F, H, E)$ is assigned to the variable B. Finally, we call the function Blattner, which takes as input $\mathfrak{g}$, the basis $\mathbf{Z} := (X_1, \ldots, X_k, Y_1, \ldots, Y_n)$, the codimension n, and the degree up to which the formal power series coefficients should be computed (here 3). The output of Blattner is a pair consisting of the (truncated) image of $\mathbf{Z}$, and the Weyl algebra generated by the x_i and the ∂_i. The latter is constructed so that one can actually calculate with the images of the basis elements (see Chapter 5). For implementation details of Blattner see Appendix B.

In order to analyse the image of $\phi_{\mathbf{Y}}$, we introduce the *concatenation of multi-indices*. Let a be a non-negative integer, $\mathbf{r}, \mathbf{r}', \mathbf{r}'' \in \mathbb{N}^a$, and $i \in \mathbb{N}$. Then the statement $\mathbf{r} = \mathbf{r}' +\!\!\!+_i \mathbf{r}''$ means

(1) $\mathbf{r}'_j = 0$ for $a \geq j > i$, and
(2) $\mathbf{r}''_j = 0$ for $1 \leq j < i$, and
(3) $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$.

We write $\mathbf{r} = \mathbf{r}' +\!\!\!+ \mathbf{r}''$ if there exists an i such that $\mathbf{r} = \mathbf{r}' +\!\!\!+_i \mathbf{r}''$. Whenever we write $\mathbf{r}' +\!\!\!+ \mathbf{r}''$, we assume implicitly that this is a valid expression. Thus, $+\!\!\!+$ becomes a binary operation with a restricted domain in $\mathbb{N}^a \times \mathbb{N}^a$. Whenever both formulas are valid, we have
$$(\mathbf{r}' +\!\!\!+ \mathbf{r}'') +\!\!\!+ \mathbf{r}''' = \mathbf{r}' +\!\!\!+ (\mathbf{r}'' +\!\!\!+ \mathbf{r}''');$$
therefore brackets can, and will, be left out. For example, if $a = 3$, then
$$(1, 2, 0) +\!\!\!+ (0, 4, 1) +\!\!\!+ (0, 0, 1)$$
is a valid expression equal to $(1, 6, 2)$, but $(1, 0, 2) +\!\!\!+ (0, 1, 0)$ is an invalid expression.

With respect to an ordered basis $\mathbf{Z} = (Z_1, \ldots, Z_a)$ of a Lie algebra $\mathfrak{l}$, define the linear map $L : U(\mathfrak{l}) \to \mathfrak{l}$ by
$$L\left(\sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{Z}^{\mathbf{m}}\right) := \sum_{i=1}^{a} c_{\mathbf{e}_i} Z_i;$$

that is, L maps an element of $U(\mathfrak{l})$ to its 'linear part'. In the remainder of this chapter, we will frequently encounter linear parts of elements of the form $\mathbf{Z}^{\mathbf{m}'}Z_j\mathbf{Z}^{\mathbf{m}''}$, where $\mathbf{m}'\!+\!\mathbf{m}'' = \mathbf{m}$. If $\mathbf{m}'\!+\!_j\mathbf{m}'' = \mathbf{m}$, then $\mathbf{Z}^{\mathbf{m}'}Z_j\mathbf{Z}^{\mathbf{m}''}$ is a PBW-monomial, hence $L(\mathbf{Z}^{\mathbf{m}'}Z_j\mathbf{Z}^{\mathbf{m}''}) = 0$ unless $\mathbf{m}' = \mathbf{m}'' = 0$. On the other hand, if $\mathbf{m}'\!+\!_i\mathbf{m}'' = \mathbf{m}$ implies $i > j$, then Z_j is not 'in the right place', and we have

$$L(\mathbf{Z}^{\mathbf{m}'}Z_j\mathbf{Z}^{\mathbf{m}''}) = \sum_{i,\mathbf{r}',\mathbf{r}'':i>j,\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}'} L(\mathbf{Z}^{\mathbf{r}'}[Z_i,Z_j]\mathbf{Z}^{\mathbf{r}''+\mathbf{m}''}).$$

Similarly, if $\mathbf{m}'\!+\!_i\mathbf{m}'' = \mathbf{m}$ implies $i < j$, then we find

$$L(\mathbf{Z}^{\mathbf{m}'}Z_j\mathbf{Z}^{\mathbf{m}''}) = \sum_{i,\mathbf{r}',\mathbf{r}'':i<j,\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}''} L(\mathbf{Z}^{\mathbf{m}'+\mathbf{r}'}[Z_j,Z_i]\mathbf{Z}^{\mathbf{r}''}).$$

In the remainder of this section, we assume that K is endowed with an Archimedean valuation $|\cdot|$, normalized such that $|n| = n$ for $n \in \mathbb{N}$. We call an element $\sum_{\mathbf{m}\in\mathbb{N}^n} c_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}$ *convergent* if there exist real numbers $C, D > 0$ such that $|c_{\mathbf{m}}| \leq DC^{|\mathbf{m}|}$ for all $\mathbf{m} \in \mathbb{N}^n$ [28], where $|\mathbf{m}|$ denotes the total degree of $\mathbf{m}$. Note that in the real or complex setting this notion of convergence is usually referred to as *absolute convergence*. Elements of $\hat{\mathfrak{D}}$ with convergent coefficients and Lie algebras of such derivations are also called convergent, and so is a realization whose image is convergent. The set of convergent power series is a subalgebra of $K[[\mathbf{x}]]$, and its automorphism group consists of the automorphisms of $K[[\mathbf{x}]]$ sending each x_i to a convergent power series. Such coordinate changes are called convergent.

PROPOSITION 2.3.3. *The realization $\phi_{\mathbf{Y}}$ is convergent.*

PROOF. Let $\|\cdot\| : \mathfrak{g} \to \mathbb{R}_{\geq 0}$ be a norm on $\mathfrak{g}$ compatible with the valuation, and choose $C > 0$ such that

$$\|\operatorname{ad}(Y_i)X\| \leq C\|X\|,$$

for all $i = 1,\ldots,n$ and all $X \in \mathfrak{g}$. Then, if $\mathbf{m} = \mathbf{m}'+\mathbf{m}''$, we have

$$\tag{2} \|L(\mathbf{Y}^{\mathbf{m}'}X\mathbf{Y}^{\mathbf{m}''})\| \leq C^{|\mathbf{m}|}|\mathbf{m}|!\|X\|$$

for all $X \in \mathfrak{g}$. To see this, proceed by induction on $|\mathbf{m}|$. For $|\mathbf{m}| = 0$ the statement is trivial. Suppose that the statements holds for $|\mathbf{m}| = d$, and let $\mathbf{m}$ be of total degree $d + 1$. Then, for $X \in \mathfrak{k}$, we have

$$
\begin{aligned}
\|L(\mathbf{Y}^{\mathbf{m}'}X\mathbf{Y}^{\mathbf{m}''})\| &= \|\sum_{\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}'} L(\mathbf{Y}^{\mathbf{r}'}[Y_i,X]\mathbf{Y}^{\mathbf{r}''+\mathbf{m}''})\| \\
&\leq \sum_{\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}'} \|L(\mathbf{Y}^{\mathbf{r}'}[Y_i,X]\mathbf{Y}^{\mathbf{r}''+\mathbf{m}''})\| \\
&\leq \sum_{\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}'} C^d d!\|[Y_i,X]\| \\
&\leq \sum_{\mathbf{r}'+\mathbf{e}_i+\mathbf{r}''=\mathbf{m}'} C^{d+1}d!\|X\| \\
&\leq (d+1)C^{d+1}d!\|X\| \\
&= C^{d+1}(d+1)!\|X\|.
\end{aligned}
$$

In the third step the induction hypothesis is used, and the fifth step uses the fact that there are at most $d+1$ terms. A similar reasoning applies when X is replaced by Y_i; this proves (2). In particular, there exists a constant $D > 0$ such that

$$|\chi_i(\mathbf{Y^m}X)| \le DC^{|\mathbf{m}|}|\mathbf{m}|!\|X\|$$

for all $X \in \mathfrak{g}, \mathbf{m} \in \mathbb{N}^n, i \in \{1,\dots,n\}$. Now use the upper bound

$$(3) \qquad |\mathbf{m}|! \le n^{|\mathbf{m}|}\mathbf{m}!$$

to find

$$\left|\frac{\chi_i(\mathbf{Y^m}X)}{\mathbf{m}!}\right| \le D(Cn)^{|\mathbf{m}|}\|X\|,$$

from whence the lemma follows using the Realization Formula.

$\square$

In view of this proposition, the question arises of whether the classification of convergent finite-dimensional transitive Lie algebras of vector fields up to convergent coordinate changes is the same as the classification of formal finite-dimensional transitive Lie algebras up to formal coordinate changes. The following theorem, whose proof reviews Blattner's proof of the uniqueness part of the Realization Theorem, combined with Proposition 2.3.3, shows that it is.

THEOREM 2.3.4. *Let ψ be a realization of $(\mathfrak{g}, \mathfrak{k})$ with convergent coefficients. Then the unique formal coordinate change θ satisfying $\theta \circ \psi(X) = \phi_{\mathbf{Y}}(X) \circ \theta$ for all $X \in \mathfrak{g}$ is convergent.*

The proof of this theorem uses the following lemma.

LEMMA 2.3.5. *Let f denote the convergent power series*

$$\sum_{\mathbf{m}\in\mathbb{N}^n} \mathbf{x^m}$$

and define

$$Y := f \sum_{i=1}^{n} \partial_i.$$

Then we have for $m \in \mathbb{N}$

$$(Y^m(f))(0) = \prod_{k=1}^{m}((n+1)k - 1) \le (n+1)^m m!$$

PROOF. We write f as

$$\frac{1}{\prod_{j=1}^{n}(1-x_j)},$$

and proceed by induction to show that

$$Y^m(f) = \sum_{\mathbf{s}\in\mathbb{N}^n, |\mathbf{s}|=m} \frac{a_{m,\mathbf{s}}}{\prod_{j=1}^{n}(1-x_j)^{m+s_j+1}}$$

for certain coefficients $a_{m,\mathbf{s}}$ satisfying

$$\sum_{\mathbf{s}\in\mathbb{N}^n, |\mathbf{s}|=m} a_{m,\mathbf{s}} = \prod_{k=1}^{m}((n+1)k - 1).$$

This is clearly the case for $m = 0$; assume that it holds for m and compute

$$Y^{m+1}(f) = Y \sum_{\mathbf{s} \in \mathbb{N}^n, |\mathbf{s}|=m} \frac{a_{m,\mathbf{s}}}{\prod_{j=1}^n (1-x_j)^{m+s_j+1}}$$

$$= \sum_{\mathbf{s} \in \mathbb{N}^n, |\mathbf{s}|=m} \sum_{i=1}^n \frac{(m+s_i+1)a_{m,\mathbf{s}}}{\prod_{j=1}^n (1-x_j)^{(m+1)+s_j+1+\delta_{ij}}}$$

$$= \sum_{\mathbf{s} \in \mathbb{N}^n, |\mathbf{s}|=m+1} \frac{a_{m+1,\mathbf{s}}}{\prod_{j=1}^n (1-x_j)^{(m+1)+s_j+1}},$$

where the $a_{m+1,\mathbf{s}}$ sum up to

$$\sum_{\mathbf{s} \in \mathbb{N}^n, |\mathbf{s}|=m} \sum_{i=1}^n (m+s_i+1)a_{m,\mathbf{s}} = \sum_{\mathbf{s} \in \mathbb{N}^n, |\mathbf{s}|=m} (nm+m+n)a_{m,\mathbf{s}} = \prod_{k=1}^{m+1} ((n+1)k-1),$$

from whence the lemma follows. $\qquad\square$

PROOF OF THEOREM 2.3.4. We follow Blattner's proof of the uniqueness part of the Realization Theorem. To this end, recall the notation of the proof of Theorem 2.3.1, and view $K[[\mathbf{x}]]$ as a $U(\mathfrak{g})$-module through $\phi_{\mathbf{Y}}$; it is isomorphic to $A = \text{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), K)$. The latter module, together with the $U(\mathfrak{k})$-module homomorphism $\gamma : \phi \mapsto \phi(1)$, $A \to K$, has the following universal property: if V is any $U(\mathfrak{g})$-module, and $\mu : V \to K$ is a homomorphism of $\mathfrak{k}$-modules, then there exists a unique homomorphism $\nu : V \to A$ of $\mathfrak{g}$-modules such that $\gamma \circ \nu = \mu$. Indeed, this homomorphism ν is defined by

$$(\nu v)u := \mu(uv)$$

for $u \in U(\mathfrak{g})$ and $v \in V$. It is straightforward to check that

$$(\beta\alpha^*)\nu v = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\mu(\mathbf{Y}^{\mathbf{m}}v)}{\mathbf{m}!}\mathbf{x}^{\mathbf{m}}.$$

where the action of $\mathfrak{g}$ on V has been extended to an action of $U(\mathfrak{g})$ on V.

Now take $V = K[[\mathbf{x}]]$ with $\mathfrak{g}$-action through ψ. The map γ corresponds, under the isomorphism $\beta \circ \alpha^*$, to the map $\sigma : f \mapsto f(0)$, $K[[\mathbf{x}]] \to K$, and σ is a homomorphism of $\mathfrak{k}$-modules from both $(K[[\mathbf{x}]], \phi_{\mathbf{Y}})$ and $(K[[\mathbf{x}]], \psi)$. Hence, the map $\theta : K[[\mathbf{x}]] \to K[[\mathbf{x}]]$ defined by

$$\theta(h) := \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{(\psi(\mathbf{Y}^{\mathbf{m}})h)(0)}{\mathbf{m}!}\mathbf{x}^{\mathbf{m}}, \ h \in K[[\mathbf{x}]],$$

where ψ has been extended to a homomorphism $U(\mathfrak{g}) \to \text{End}_K(K[[\mathbf{x}]])$, has the property that $\theta \circ \psi(X) = \phi_{\mathbf{Y}}(X) \circ \theta$ for all $X \in \mathfrak{g}$. Blattner shows that θ is an automorphism of $K[[\mathbf{x}]]$, or rather: that the corresponding map $A \to A$ is an automorphism of A with respect to the multiplication on A.

Summarizing, we find that for $h \in K[[\mathbf{x}]]$ and $\mathbf{m} \in \mathbb{N}^n$ the coefficient of $\mathbf{x}^{\mathbf{m}}$ in θh equals

$$\frac{1}{\mathbf{m}!}(\psi(Y_1)^{m_1} \cdots \psi(Y_n)^{m_n}h)(0).$$

Write

$$\psi(Y_l) = \sum_{i=1}^{n} \sum_{\mathbf{m} \in \mathbb{N}} c_{l,i,\mathbf{m}} \mathbf{x}^{\mathbf{m}} \partial_i;$$

as all $\psi(Y_l)$ are convergent, we may choose real numbers $C, D > 0$ such that

$$|c_{l,i,\mathbf{m}}| \le DC^{|\mathbf{m}|}$$

for all $l, i = 1, \ldots, n$ and $\mathbf{m} \in \mathbb{N}^n$. Now apply Lemma 2.3.5 to find

$$|(\psi(Y_1)^{m_1} \cdots \psi(Y_n)^{m_n} x_i)(0)| \le (CD(n+1))^{|\mathbf{m}|} |\mathbf{m}|!$$

Hence, using (3), we find that the coefficient of $\mathbf{x}^{\mathbf{m}}$ in θx_i has valuation at most

$$(CD(n+1))^{|\mathbf{m}|} |\mathbf{m}|!/\mathbf{m}! \le (CD(n+1)n)^{|\mathbf{m}|},$$

so that θx_i is a convergent power series. □

2.4. Realizations with Polynomial Coefficients

We will use $\phi_{\mathbf{Y}}$ to prove Theorem 2.2.3, as well as some variants. To this end, recall the notion of a polynomial transitive realization of page 8. Many results on such realizations can be found in the literature; let me mention a few.

(1) In [**62**], such realizations are constructed of pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is a complex classical simple Lie algebra, and $\mathfrak{k}$ is a maximal parabolic subalgebra.
(2) In [**26**], this is done for general complex semisimple Lie algebras $\mathfrak{g}$ and certain maximal parabolic $\mathfrak{k}$. The polynomials occurring have degree at most 4. In this article, some homogeneous rational realizations are derived as well.
(3) The article [**27**] contains the following theorem: if $\mathfrak{k}$ has a vector space complement in $\mathfrak{g}$ that is a subalgebra acting nilpotently on the complex Lie algebra $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{k})$ has a polynomial transitive realization.
(4) In [**53**], transitive realizations are given of pairs $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is complex simple and $\mathfrak{k}$ is any maximal parabolic subalgebra $\mathfrak{k}$. It is proved that for the classical cases, the total degree of the polynomials occurring can be bounded by 4, whereas for the exceptional cases, polynomials of higher degree may be necessary.

All constructions above make use of an $\text{ad}_{\mathfrak{g}}$-nilpotent complementary subalgebra $\mathfrak{m}$ of $\mathfrak{k}$, which is used as a coordinate chart for the homogeneous space G/H of the corresponding Lie groups. The Baker-Campbell-Hausdorff formula is used to compute explicitly the vector fields on G/H induced by $\mathfrak{g}$, in the coordinates provided by $\mathfrak{m}$. In this section we slightly generalize Theorem 5 of [**27**], using the Realization Formula instead of working in a homogeneous space. However, we first prove Theorem 2.2.3; this is essentially Gradl's theorem, but for arbitrary fields of characteristic 0 instead of $\mathbb{C}$ or $\mathbb{R}$.

PROOF OF THEOREM 2.2.3. As $\mathfrak{m}$ acts nilpotently on itself, we can choose a basis $Y_1, \ldots, Y_n$ of $\mathfrak{m}$ such that

$$[\mathfrak{m}, Y_i] \subseteq \langle Y_{i+1}, \ldots, Y_n \rangle$$

for all $i = 1, \ldots, n$. The algebra $\mathfrak{m}$ also acts nilpotently on $\mathfrak{g}/\mathfrak{m}$; hence, we can choose a basis $X_1, \ldots, X_k$ of $\mathfrak{k}$ such that

$$[\mathfrak{m}, X_i] \subseteq \langle X_{i+1}, \ldots, X_k \rangle + \mathfrak{m}$$

for all $i = 1, \ldots, k$. For $i = 1, \ldots, k + n$, define

$$Z_i := \begin{cases} X_i & \text{if } i \leq k, \text{ and} \\ Y_{i-k} & \text{if } i > k. \end{cases}$$

By induction on j, we will show that for all j,

$$L(\mathbf{Y}^{\mathbf{r}'} Z_j \mathbf{Y}^{\mathbf{r}''}) = 0 \text{ for } |\mathbf{r}' + \mathbf{r}''| \text{ sufficiently large.}$$

Suppose that this is true for all $l > j$, and consider the expression $\mathbf{Y}^{\mathbf{r}'} Z_j \mathbf{Y}^{\mathbf{r}''}$, where $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$. Applying the PBW rewriting rules, we find that $L(\mathbf{Y}^{\mathbf{r}'} Z_j \mathbf{Y}^{\mathbf{r}''})$ is a linear combination of terms of the form

$$L(\mathbf{Y}^{\mathbf{t}'} [Y_i, Z_j] \mathbf{Y}^{\mathbf{t}''})$$

with $\mathbf{t}' + \mathbf{e}_i + \mathbf{t}'' = \mathbf{r}$, so that $\mathbf{t} := \mathbf{t}' + \mathbf{t}''$ has total degree $|\mathbf{t}| = |\mathbf{r}| - 1$. As $\mathrm{ad}_\mathfrak{g} Y_i$ has a lower triangular matrix with respect to the basis $\mathbf{Z}$, the Lie brackets can be expanded to obtain a linear combination of terms of the form $\mathbf{Y}^{\mathbf{t}'} Z_l \mathbf{Y}^{\mathbf{t}''}$ with $l > j$. By the induction hypothesis, for $|\mathbf{t}|$ large enough, each of these terms has zero linear part. Hence so does the original expression for sufficiently large $|\mathbf{r}|$.

Taking $\mathbf{r}'' = 0$, we find that for all $X \in \mathfrak{g}$:

$$L(\mathbf{Y}^{\mathbf{r}} X) = 0 \text{ for large } |\mathbf{r}|.$$

It follows that $\phi_{\mathbf{Y}}$ is a polynomial realization. $\qquad\qquad\square$

EXAMPLE 2.4.1. This theorem proves the existence of polynomial transitive realizations of pairs $(\mathfrak{g}, \mathfrak{p}_{\Pi_0})$, where $\mathfrak{g}$ is semisimple and $\mathfrak{p}_{\Pi_0}$ is the parabolic subalgebra defined on page 7. Indeed, in the notation of that page, the subspace $\bigoplus_{\alpha \in \Delta \setminus (\Delta_0 \cup \Delta_+)} \mathfrak{g}_\alpha$ is a vector space complement of $\mathfrak{p}_{\Pi_0}$, as well as a subalgebra of $\mathfrak{g}$ acting nilpotently on the latter.

Realizations of such pairs can be computed explicitly by my `GAP`-program. Here is a printout of a `GAP`-session in which a realization of the pair $(\mathfrak{sl}_3, \mathfrak{p}_1)$ is computed, where $\mathfrak{p}_1$ is the maximal parabolic subalgebra corresponding to the first node of the Dynkin diagram of A_2; see Section 1.2.

```
gap> g:=SimpleLieAlgebra("A",2,Rationals);;
gap> Y:=Basis(g,GeneratorsOfAlgebra(g){[2,4,5,6,7,8,1,3]});;
gap> Blattner(g,Y,2,3)[1];
[ [(-1)*x_1*D_2], [(-1)*x_1*x_2*D_2+(-1)*x_1^2*D_1], [(-1)*x_2*D_1],
  [(-1)*x_1*x_2*D_1+(-1)*x_2^2*D_2], [(-2)*x_1*D_1+(-1)*x_2*D_2],
  [(1)*x_1*D_1+(-1)*x_2*D_2], [(1)*D_1], [(1)*D_2] ]
```

The default basis of a split Lie algebra of type A_2 in `GAP` is ordered as follows: first the positive root vectors, then the negative root vectors, and finally the Chevalley basis of the Cartan subalgebra. This explains the reordering in the second line of the session. The output is a familiar Lie algebra from Lie's list; see Table 1.

We can slightly generalize Theorem 2.2.3 to the following theorem.

THEOREM 2.4.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{k}, \mathfrak{h}, \mathfrak{n}$ be subalgebras such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}$ as vector spaces. Assume that both $\mathfrak{n}$ and $\mathfrak{k}$ are invariant under $\mathrm{ad}_\mathfrak{g} \mathfrak{h}$, that $\mathfrak{h}$ is nilpotent, and that $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ has a polynomial transitive realization.*

Note that we recover Theorem 2.2.3 in taking $\mathfrak{h} = 0$.

PROOF. As $\mathfrak{n}$ acts nilpotently on itself, we may choose a basis $Z_1, \ldots, Z_c$ of $\mathfrak{n}$ with the property that

$$[\mathfrak{n}, Z_i] \subseteq \langle Z_{i+1}, \ldots, Z_c \rangle$$

for all i. Similarly, we may choose a basis $Y_1, \ldots, Y_b$ of $\mathfrak{h}$ such that

$$[\mathfrak{h}, Y_i] \subseteq \langle Y_{i+1}, \ldots, Y_b \rangle$$

for all i. Finally, $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}/(\mathfrak{h} \oplus \mathfrak{n})$, so that one can choose a basis $X_1, \ldots, X_a$ of $\mathfrak{k}$ such that

$$[\mathfrak{n}, X_i] \subseteq \langle X_{i+1}, \ldots, X_a \rangle \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

Define, for $i = 1, \ldots, a + b + c$,

$$U_i := \begin{cases} X_i & \text{if } i \leq a, \\ Y_{i-a} & \text{if } a < i \leq a + b, \text{ and} \\ Z_{i-a-b} & \text{if } a + b < i. \end{cases}$$

We claim that, for all $j = 1, \ldots, a + b + c$,

$$L(\mathbf{Y^r Z^s} U_j) \in \mathfrak{k}$$

for $|\mathbf{r}| + |\mathbf{s}|$ sufficiently large, and we proceed by obtaining similar results for expressions that emerge when applying the PBW rewriting rules to $\mathbf{Y^r Z^s} U_j$.

First consider the expression $\mathbf{Y^{r'}} X \mathbf{Y^{r''} Z^s}$, where $X \in \mathfrak{k}$ and $\mathbf{r'} + \mathbf{r''} = \mathbf{r}$. Since $\mathfrak{k} + \mathfrak{h}$ is a subalgebra, this reduces to an expression in which all PBW-monomials end with $\mathbf{Z^s}$. Hence, if $\mathbf{s} \neq 0$, then one has

$$L(\mathbf{Y^{r'}} X \mathbf{Y^{r''} Z^s}) = 0.$$

For the case $\mathbf{s} = 0$, induction on $|\mathbf{r}|$ shows that

$$L(\mathbf{Y^{r'}} X \mathbf{Y^{r''}}) \in \mathfrak{k}.$$

Indeed, this is obvious for $|\mathbf{r}| = 0$. Assume that it holds for $|\mathbf{r}| = d$, let $\mathbf{r}$ be of total degree $d + 1$, and split $\mathbf{r} = \mathbf{r'} + \mathbf{r''}$ in any manner. Then $L(\mathbf{Y^{r'}} X \mathbf{Y^{r''}})$ equals a linear combination of terms $L(\mathbf{Y^{t'}} [Y_i, X] \mathbf{Y^{t''}})$ with $\mathbf{t'} + \mathbf{e}_i + \mathbf{t''} = \mathbf{r}$. As $[Y_i, X] \in \mathfrak{k}$, the induction hypothesis applies to each of these terms.

Next consider the expression $\mathbf{Y^{r'}} Y_j \mathbf{Y^{r''} Z^s}$, where $\mathbf{r'} + \mathbf{r''} = \mathbf{r}$. We claim that, for $|\mathbf{r}| + |\mathbf{s}|$ large enough, its linear part is zero. Again, the case $\mathbf{s} \neq 0$ is easy, so assume that $\mathbf{s} = 0$. But this situation is handled by the proof of Theorem 2.2.3.

Finally, consider the expression $\mathbf{Y^r Z^{s'}} U_j \mathbf{Z^{s''}}$, where $\mathbf{s'} + \mathbf{s''} = \mathbf{s}$. By induction on j, we shall see that its linear part is in $\mathfrak{k}$ for $|\mathbf{r}| + |\mathbf{s}|$ sufficiently large. Suppose that this is the case for all $l > j$. Under the PBW rewriting rules, $L(\mathbf{Y^r Z^{s'}} U_j \mathbf{Z^{s''}})$ is seen to equal a linear combination of terms $L(\mathbf{Y^r Z^{t'}} [Z_i, U_j] \mathbf{Z^{t''}})$ with $\mathbf{t'} + \mathbf{e}_i + \mathbf{t''} = \mathbf{s}$, and possibly a term $L(\mathbf{Y^r} U_j \mathbf{Z^s})$. The latter only occurs if $j \leq a + b$, and the preceding two paragraphs show that its linear part is indeed in $\mathfrak{k}$, if $|\mathbf{r}| + |\mathbf{s}|$ is large enough. As for the former terms, they are linear combinations of terms $L(\mathbf{Y^r Z^{t'}} U_l \mathbf{Z^{t''}})$ with $l > j$, because Z_i acts nilpotently. The induction hypothesis applies, and this concludes the proof of our claim.

By taking $\mathbf{s}'' = 0$ in the conclusion of the previous paragraph, we find that the linear part of $\mathbf{Y^r Z^s} U_j$ is an element of $\mathfrak{k}$ for $|\mathbf{r} + \mathbf{s}|$ sufficiently large. In particular, the coefficients of the PBW-monomials Y_i and Z_i in $\mathbf{Y^r Z^s} U_j$ are zero for $|\mathbf{r} + \mathbf{s}|$ sufficiently large. Using the Realization Formula, we find that $\phi_{\mathbf{Y},\mathbf{Z}}$ is polynomial. $\square$

EXAMPLE 2.4.3. Let $\mathfrak{g}$ be the Lie algebra spanned by E, H, F, I, X, Y, and Z where E, H, F are the Chevalley basis of an $\mathfrak{sl}_2$ that commutes with I, and further the only non-zero relations are given by

$$[H, X] = X, \qquad [H, Y] = -Y, \qquad [F, X] = Y, \qquad [E, Y] = X,$$
$$[I, X] = X, \qquad [I, Y] = Y, \qquad [I, Z] = 2Z, \qquad [X, Y] = Z.$$

Then $\mathfrak{k} = \langle E, H, F \rangle_K$, $\mathfrak{h} = \langle I \rangle_K$, and $\mathfrak{n} = \langle X, Y, Z \rangle_K$ satisfy the hypotheses of Theorem 2.4.2. In Example 2.4.3, the polynomial realization $\phi_{(I,X,Y,Z)}$ of the pair $(\mathfrak{g}, \mathfrak{k})$ is computed in GAP; it is determined by

$$E \mapsto -x_3 \partial_2 + \frac{1}{2} x_3^2 \partial_4, \qquad H \mapsto -x_2 \partial_2 + x_3 \partial_3,$$
$$F \mapsto -x_2 \partial_3 + \frac{1}{2} x_2^2 \partial_4, \qquad I \mapsto -x_2 \partial_2 - x_3 \partial_3 - x_4 \partial_4 + \partial_1,$$
$$X \mapsto -x_3 \partial_4 + \partial_2, \qquad Y \mapsto \partial_3, \text{ and}$$
$$Z \mapsto \partial_4.$$

In a similar fashion, one can prove the following theorem.

THEOREM 2.4.4. *Let $\mathfrak{g}$ be a split semisimple Lie algebra. Then $(\mathfrak{g}, 0)$ has a polynomial transitive realization.*

PROOF. Consider the ordered basis $H_1, \ldots, H_l, X_1, \ldots, X_m, Y_1, \ldots, Y_m$, where the H_i span a Cartan subalgebra, the X_i are positive root vectors, and the Y_i are negative root vectors. One can prove that $\phi_{(\mathbf{H}, \mathbf{X}, \mathbf{Y})}$ is a polynomial realization. $\square$

REMARK 2.4.5. In the setting of this proof, one also has

$$L(\mathbf{H^r X^{s'}} Y_j \mathbf{X^{s''} Y^t}) = 0,$$

for large $\|\mathbf{r} + \mathbf{s} + \mathbf{t}\|$, where $\mathbf{s} = \mathbf{s}' + \mathbf{s}''$. This remark will be useful in the proof of Theorem 2.5.3.

2.5. Realizations with Coefficients in E

We want to use the Realization Formula to prove Theorem 2.2.4. For a formal power series to be an element of $E^{(n)}$, it must satisfy linear ordinary differential equations with constant coefficients in each of its variables. Therefore, we investigate the behaviour of a coefficient in $\phi_{\mathbf{Y}}$ under differential operators.

LEMMA 2.5.1. *Let $p(t) \in K[t]$ be a univariate polynomial. The formal power series*

$$f := \sum_{\mathbf{m}} \chi_i(\mathbf{Y^m} X) \frac{\mathbf{x^m}}{\mathbf{m!}},$$

occurring in the Realization Formula, satisfies

$$p(\partial_l) f = \sum_{\mathbf{m}} \chi_i(\mathbf{Y^{m'}} P(Y_l) \mathbf{Y^{m''}} X) \frac{\mathbf{x^m}}{\mathbf{m!}},$$

where, for each $\mathbf{m} \in \mathbb{N}^n$, *one partition* $\mathbf{m} = \mathbf{m}' \mathbin{+\!\!+}_l \mathbf{m}''$ *is chosen.*

PROOF. It suffices to prove this in the case where $p(t)$ is a monomial; taking linear combinations on both sides then yields the result. Hence, compute

$$
\begin{aligned}
(\partial_l)^s f &= \sum_{\mathbf{m},\, m_l \geq s} \chi_i(\mathbf{Y^m} X) m_l(m_l - 1) \cdot \ldots \cdot (m_l - s + 1) \frac{\mathbf{x}^{\mathbf{m} - s\mathbf{e}_l}}{\mathbf{m}!} \\
&= \sum_{\mathbf{m},\, m_l \geq s} \chi_i(\mathbf{Y^m} X) \frac{\mathbf{x}^{\mathbf{m} - s\mathbf{e}_l}}{(\mathbf{m} - s\mathbf{e}_l)!} \\
&= \sum_{\mathbf{m}} \chi_i(\mathbf{Y}^{\mathbf{m} + s\mathbf{e}_l} X) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!},
\end{aligned}
$$

which was to be proved. $\qquad\square$

In the setting of Theorem 2.2.4, we can construct polynomials in the ∂_l that annihilate the coefficients occurring in $\phi_{\mathbf{Y}}(\mathfrak{g})$.

PROOF OF THEOREM 2.2.4. Choose a basis $X_1, \ldots, X_k$ of $\mathfrak{k}$, and choose $Y_l \in \mathfrak{g}_l \setminus \mathfrak{g}_{l-1}$ for $l = 1, \ldots, n$. We will prove that $\phi_{\mathbf{Y}}$ has coefficients in $E^{(n)}$. To this end, consider the linear maps $A_l : \mathfrak{g} \to \mathfrak{g}$ for $l = 1, \ldots, n$ determined by

$$
\begin{aligned}
A_l X_j &= [Y_l, X_j] \text{ for all } j, \\
A_l Y_j &= [Y_l, Y_j] \text{ for all } j < l, \text{ and} \\
A_l Y_j &= 0 \text{ for all } j \geq l.
\end{aligned}
$$

We claim that, for all $\mathbf{m} \in \mathbb{N}^n$,

$$
L(\mathbf{Y^m} X) = L(\mathbf{A^m} X),
$$

where $\mathbf{A^m} := A_1^{m_1} \cdots A_n^{m_n}$. Indeed, let $\mathbf{m} = \mathbf{m}' + \mathbf{e}_l$, and calculate

$$
L(\mathbf{Y^m} X_j) = L(\mathbf{Y}^{\mathbf{m}'} Y_l X_j) = L(\mathbf{Y}^{\mathbf{m}'} X_j Y_l) + L(\mathbf{Y}^{\mathbf{m}'} [Y_l, X_j]).
$$

Now $\mathbf{Y}^{\mathbf{m}'} X_j$ is an element of $U(\mathfrak{g}_l)$, so that the PBW-monomials occurring in it contain only $X_1, \ldots, X_k, Y_1, \ldots, Y_l$. Hence, $L(\mathbf{Y}^{\mathbf{m}'} X_j Y_l) = 0$, so that

$$
L(\mathbf{Y^m} X_j) = L(\mathbf{Y}^{\mathbf{m}'} (A_l(X_j))).
$$

For the same reason, this holds if one replaces X_j by Y_j with $j < l$. Finally, if $j \geq l$, then $\mathbf{Y^m} Y_j$ is a PBW-monomial, so that $L(\mathbf{Y^m} Y_j) = 0$. By induction, this proves the claim.

Let p_l be the minimal polynomial of A_l. Application of Lemma 2.5.1 shows that

$$
p_l(\partial_l) f = 0
$$

for f as in that lemma. Hence, the realization $\phi_{\mathbf{Y}}$ has coefficients in $E^{(n)}$, as stated. $\quad\square$

Olivier Mathieu noted the following corollary.

COROLLARY 2.5.2. *Let* $\mathfrak{g}$ *be a finite-dimensional soluble Lie algebra, and let* $\mathfrak{k}$ *be any subalgebra; let* n *be the codimension of* $\mathfrak{k}$ *in* $\mathfrak{g}$. *Then* $(\mathfrak{g}, \mathfrak{k})$ *has a transitive realization with coefficients in* $E^{(n)}$.

PROOF. Set $\mathfrak{g}_0 := \mathfrak{k}$. Suppose that we have found a sequence $\mathfrak{g}_0 \subsetneq \ldots \subsetneq \mathfrak{g}_l \subsetneq \mathfrak{g}$ of subalgebras such that $\dim \mathfrak{g}_i = \mathfrak{k} + i$. As $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}_l} \mathfrak{g}_l$ is a soluble Lie algebra of linear transformations on $\mathfrak{g}/\mathfrak{g}_l$, Lie's theorem ([**35**], Chapter II) applies, and there exists a $Y \in \mathfrak{g}$ such that $\mathfrak{g}_{l+1} := \mathfrak{g}_l \oplus KY$ is a subalgebra. We thus find a sequence

$$\mathfrak{k} = \mathfrak{g}_0 \subsetneq \ldots \subsetneq \mathfrak{g}_n = \mathfrak{g}$$

of subalgebras, and Theorem 2.2.4 applies. $\qquad\square$

Another class of Lie algebras with coefficients in $E^{(n)}$ is presented in the following theorem.

THEOREM 2.5.3. *Let $\mathfrak{k}_1, \mathfrak{k}_2$ be isomorphic split semisimple Lie algebras, and set $\mathfrak{g} := \mathfrak{k}_1 \oplus \mathfrak{k}_2$. Let $\mathfrak{k}$ be the diagonal subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k})$ has a transitive realization with coefficients in $E^{(\dim \mathfrak{k})}$.*

PROOF. For $i = 1, 2$, let $\mathbf{H}_i := (H_{i1}, \ldots, H_{il})$ be a basis of a split Cartan subalgebra of $\mathfrak{k}_i$, and let $\mathbf{E}_i := (E_{i1}, \ldots, E_{im})$ and $\mathbf{F}_i := (F_{i1}, \ldots, F_{im})$ be lists of positive and negative root vectors, respectively, such that $\mathfrak{k}$ is spanned by $H_{1j} + H_{2j}$ $(j = 1, \ldots, l)$, and $E_{1j} + E_{2j}, F_{1j} + F_{2j}$ $(j = 1, \ldots, m)$. Now $\mathbf{Y} = (\mathbf{H}_1, \mathbf{E}_1, \mathbf{F}_1)$ is a basis of a vector space complementary to $\mathfrak{k}$ in $\mathfrak{g}$, and we claim that $\phi_{\mathbf{Y}}$ has coefficients in $E^{(l+2m)}$. Indeed, from the proof of Theorem 2.4.4 we know that for $X \in \mathfrak{k}_1$:

$$L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}} X) = 0 \text{ if } |\mathbf{r} + \mathbf{s} + \mathbf{t}| \text{ is sufficiently large.}$$

Hence, $\phi_{\mathbf{Y}}(\mathfrak{k}_1)$ has polynomial coefficients. Next, compute

$$
\begin{aligned}
L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}}(H_{1j} + H_{2j})) = & \sum_{\mathbf{r}' + \mathbf{e}_i + \mathbf{r}'' = \mathbf{r}} L(\mathbf{H}_1^{\mathbf{r}'}[H_{1i}, H_{1j} + H_{2j}]\mathbf{H}_1^{\mathbf{r}''} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}}) \\
& + \sum_{\mathbf{s}' + \mathbf{e}_i + \mathbf{s}'' = \mathbf{s}} L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}'}[E_{1i}, H_{1j} + H_{2j}]\mathbf{E}_1^{\mathbf{s}''} \mathbf{F}_1^{\mathbf{t}}) \\
& + \sum_{\mathbf{t}' + \mathbf{e}_i + \mathbf{t}'' = \mathbf{t}} L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}'}[F_{1i}, H_{1j} + H_{2j}]\mathbf{F}_1^{\mathbf{t}''}).
\end{aligned}
$$

Since the E_{1i} and F_{1i} are root vectors, the arguments of L in the last two terms are PBW-monomials. Hence, if $|\mathbf{r} + \mathbf{s} + \mathbf{t}| > 1$, then the last two sums are zero. The first sum is zero since the Cartan subalgebra is Abelian. Hence, $\phi_{\mathbf{Y}}(H_{1j} + H_{2j})$ is a polynomial vector field for all j. Compute

$$
\begin{aligned}
L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}}(E_{1j} + E_{2j})) = & \sum_{\mathbf{r}' + \mathbf{e}_i + \mathbf{r}'' = \mathbf{r}} L(\mathbf{H}_1^{\mathbf{r}'}[H_{1i}, E_{1j} + E_{2j}]\mathbf{H}_1^{\mathbf{r}''} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}}) \\
(4) \qquad & + \sum_{\mathbf{s}' + \mathbf{e}_i + \mathbf{s}'' = \mathbf{s}} L(\mathbf{H}_1^{\mathbf{r}} \mathbf{E}_1^{\mathbf{s}'}[E_{1i}, E_{1j} + E_{2j}]\mathbf{E}_1^{\mathbf{s}''} \mathbf{F}_1^{t}) \\
& + \sum_{\mathbf{t}' + \mathbf{e}_i + \mathbf{t}'' = \mathbf{t}} L(\mathbf{H}_1^{r} \mathbf{E}_1^{s} \mathbf{F}_1^{\mathbf{t}'}[F_{1i}, E_{1j} + E_{2j}]\mathbf{F}_1^{t''}).
\end{aligned}
$$

For $|\mathbf{r} + \mathbf{s} + \mathbf{t}|$ sufficiently large, the last two sums are zero; the same argument applies as for the polynomial realization of $(\mathfrak{k}_1, 0)$. A term in the first sum is zero unless $\mathbf{r}' = 0$. It follows that the first sum reduces to

$$(5) \qquad\qquad -L((A_l^{r_l} A_{l-1}^{r_{l-1}} \ldots A_1^{r_1} E_{1j}) \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^{\mathbf{t}}),$$

where $A_i = -\operatorname{ad}_{\mathfrak{g}} H_{1i}$. Again, for $|\mathbf{s} + \mathbf{t}|$ large, this is zero. Hence, the variables corresponding to the E_{1i} or to the F_{1i} appear polynomially. As for the remaining variables: let q_i be the minimal polynomial of A_i, and define $p_i(t) := t^d q_i(t)$ for some sufficiently large d. Then we find

$$L(\mathbf{H}_1^{\mathbf{r}'} p_i(H_{1i}) \mathbf{H}_1^{\mathbf{r}''} \mathbf{E}_1^{\mathbf{s}} \mathbf{F}_1^t (E_{1j} + E_{2j})) = 0,$$

for all $\mathbf{r} = \mathbf{r}' +_i \mathbf{r}''$, $\mathbf{s}$, and $\mathbf{t}$. The factor t^d in p_i is to ensure that the last two sums in (4) are zero, so that only the one in (5) remains. This, in turn, is killed by q_i.

Similarly, using Remark 2.4.5, we find that the $F_{1j} + F_{2j}$ are also realized with coefficients in $E^{(l+2m)}$.   □

To better appreciate Theorem 2.5.3, consider the following example.

EXAMPLE 2.5.4. Let $\mathfrak{g}$ be $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and let $\mathfrak{k}$ be the diagonal subalgebra. The following computations in GAP show a transitive realization of this pair.

```
gap> k:=SimpleLieAlgebra("A",1,Rationals);;
gap> g:=DirectSumOfAlgebras(k,k);;
gap> X:=GeneratorsOfAlgebra(g);;
gap> B:=BasisByGenerators(g,
  Concatenation(X{[1..3]}+X{[4..6]},X{[3,1,2]}));;
gap> Blattner(g,B,3,5)[1];
[ [(2)*x_1*D_2+(-2)*x_1^2*D_2+(4/3)*x_1^3*D_2+(-2/3)*x_1^4*D_2+
    (4/15)*x_1^5*D_2+(2)*x_2*x_3*D_2+(-1)*x_3*D_1+(-1)*x_3^2*D_3],
  [(2)*x_1*x_2*D_1+(-2)*x_1*x_2^2*D_2+(-2)*x_1*D_3+
    (2)*x_1^2*x_2*D_1+(-2)*x_1^2*x_2^2*D_2+(-2)*x_1^2*D_3+
    (4/3)*x_1^3*x_2*D_1+(-4/3)*x_1^3*x_2^2*D_2+(-4/3)*x_1^3*D_3+
    (2/3)*x_1^4*x_2*D_1+(-2/3)*x_1^4*D_3+(-4/15)*x_1^5*D_3+
    (1)*x_2*D_1+(-1)*x_2^2*D_2],
  [(-2)*x_2*D_2+(2)*x_3*D_3],
  [(-2)*x_2*D_2+(2)*x_3*D_3+(1)*D_1],
  [(2)*x_2*x_3*D_2+(-1)*x_3*D_1+(-1)*x_3^2*D_3+(1)*D_2],
  [(1)*D_3] ]
```

Denote by E_i, H_i, F_i, $i = 1, 2$ the Chevalley bases of the two copies of $\mathfrak{sl}_2$. The above formal formal power series suggest the realization ϕ determined by

$$\begin{aligned}
E_1 + E_2 &\mapsto -x_3 \partial_1 + (1 + 2x_2 x_3 - \exp(-2x_1)) \partial_2 - x_3^2 \partial_3, \\
F_1 + F_2 &\mapsto x_2 \exp(2x_1) \partial_1 - x_2^2 \exp(2x_1) \partial_2 + (1 - \exp(2x_1)) \partial_3, \\
H_1 + H_2 &\mapsto -2x_2 \partial_2 + 2x_3 \partial_3, \\
H_1 &\mapsto \partial_1 - 2x_2 \partial_2 + 2x_3 \partial_3, \\
E_1 &\mapsto -x_3 \partial_1 + (1 + 2x_2 x_3) \partial_2 - x_3^2 \partial_3, \text{ and} \\
F_1 &\mapsto \partial_3,
\end{aligned}$$

which implies

$$E_2 \mapsto -\exp(-2x_1)\partial_2,$$
$$F_2 \mapsto x_2\exp(2x_1)\partial_1 - x_2^2\exp(2x_1)\partial_2 - \exp(2x_1)\partial_3,$$
$$H_2 \mapsto -\partial_1.$$

The transition from the output of `Blattner` to this realization—for indeed, it is readily checked that ϕ is a transitive realization of the pair $(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \mathfrak{k})$—was mere guessing the closed forms of the formal power series coefficients. However, the conclusion that ϕ_{H_1,E_1,F_1} equals ϕ could proved rigorously as follows: from the proof of Theorem 2.5.3, we can extract a finite-dimensional subspace of $E^{(3)}$ in which all coefficients of ϕ_{H_1,E_1,F_1} must lie. Then we can find the closed form of these formal power series by computing their terms up to a sufficiently high degree.

In Chapters 3 and 4 we encounter ways to prove that the pair $(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ even has a polynomial realization. Indeed, Example 1.3.1 from the introduction of this thesis shows that this pair is a subpair of $(\mathfrak{sl}_4, \mathfrak{p}_1)$, which has a polynomial realization by virtue of Example 2.4.1. However, it seems impossible to find a basis $\mathbf{Y}$ of a vector space complementary to $\mathfrak{k}$ in $\mathfrak{g}$ such that $\phi_{\mathbf{Y}}$ is polynomial.

2.6. Lie's Conjecture up to Three Variables

We prove Theorem 2.2.5 by means of the techniques from the previous sections.

PROOF OF THEOREM 2.2.5. First, after possibly replacing $(\mathfrak{g}, \mathfrak{k})$ by its effective quotient, we may assume that the pair is effective. The case $\mathrm{codim}_\mathfrak{g}\, \mathfrak{k} = 1$ is a trivial instance of Theorem 2.2.4.

If $\mathrm{codim}_\mathfrak{g}\, \mathfrak{k} = 2$, then either $\mathfrak{k}$ is maximal in $\mathfrak{g}$, or not. In the latter case, Theorem 2.2.4 applies. In the former, it is immediate from Morozov's and Dynkin's theorems in Section 3.2 that there are three possibilities:

(1) $\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an Abelian two-dimensional ideal on which $\mathfrak{sl}_2$ acts irreducibly, and $\mathfrak{k} = \mathfrak{sl}_2$;
(2) $\mathfrak{g} = \mathfrak{gl}_2 \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an Abelian two-dimensional ideal on which $\mathfrak{gl}_2$ acts irreducibly, and $\mathfrak{k} = \mathfrak{gl}_2$; or
(3) $\mathfrak{g} = \mathfrak{sl}_3$, and $\mathfrak{k}$ is a maximal parabolic subalgebra.

In each of these cases, there is a subalgebra $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ as vector spaces, and such that $\mathfrak{m}$ acts nilpotently on $\mathfrak{g}$. Hence, Theorem 2.2.3 applies, and $(\mathfrak{g}, \mathfrak{k})$ has a polynomial transitive realization.

Finally, suppose that $\mathrm{codim}_\mathfrak{g}\, \mathfrak{k} = 3$. If $\mathfrak{k}$ is maximal in $\mathfrak{g}$, then one of the following holds:

(1) $\mathfrak{g}$ is simple of type A_3, B_2, C_2 and $\mathfrak{k}$ is parabolic. Then Example 2.4.1 applies, and $(\mathfrak{g}, \mathfrak{k})$ has a polynomial transitive realization.
(2) $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{m}$, where $\mathfrak{k}$ acts faithfully and irreducibly on the Abelian ideal $\mathfrak{m}$. Again, Theorem 2.2.3 yields a polynomial transitive realization.
(3) $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and $\mathfrak{k}$ is the diagonal subalgebra. But this is Example 2.5.4.

Now assume that $\mathfrak{k}$ is not maximal. First suppose that there is a maximal subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, in which $\mathfrak{k}$ is a maximal subalgebra. There are two cases to be distinguished.

First assume that $(\mathrm{codim}_{\mathfrak{g}}\,\mathfrak{l}, \mathrm{codim}_{\mathfrak{l}}\,\mathfrak{k}) = (1,2)$. Choose $Y_3 \in \mathfrak{g}\setminus\mathfrak{l}$. Let $\mathfrak{j}$ be the largest $\mathfrak{l}$-ideal in $\mathfrak{k}$. The effective quotient $(\mathfrak{l}/\mathfrak{j}, \mathfrak{k}/\mathfrak{j})$ has codimension two, and is primitive. From the above, it is clear that there is a complementary subalgebra $\mathfrak{m}/\mathfrak{j}$ that acts nilpotently on $\mathfrak{l}/\mathfrak{j}$. Let $Y_1, Y_2 \in \mathfrak{l}$ be such that $Y_1 + \mathfrak{j}, Y_2 + \mathfrak{j}$ span $\mathfrak{m}/\mathfrak{j}$, and set $\mathbf{Y} := (Y_1, Y_2, Y_3)$. Then the image of $\phi_{\mathbf{Y}}$ has coefficients in $E^{(3)}$, as can be seen as follows. For any $X \in \mathfrak{g}$, $L(Y_1^{r_1} Y_2^{r_2} Y_3^{r_3} X) = L(Y_1^{r_1} Y_2^{r_2}\,\mathrm{ad}(Y_3)^{r_3}(X))$. Like in the proof of Theorem 2.2.4 this implies that all coefficients of the realization $\phi_{\mathbf{Y}}$ are annihilated by $p(\partial_3)$, where p is the minimal polynomial of $\mathrm{ad}_{\mathfrak{g}}(Y_3)$. Next note that $L(Y_1^{r_1} Y_2^{r_2} Y_3) = 0$, and $L(Y_1^{r_1} Y_2^{r_2} X) \in \mathfrak{j}$ for sufficiently large $r_1 + r_2$, and all $X \in \mathfrak{k}$. This follows from the argument in the proof of Theorem 2.2.3. Thus we find that the variables x_1 and x_2 appear only polynomially in the coefficients of the realization.

For the case $(\mathrm{codim}_{\mathfrak{g}}\,\mathfrak{l}, \mathrm{codim}_{\mathfrak{l}}\,\mathfrak{k}) = (2,1)$, I have not found an easy argument like for the previous case. Instead of plunging into the classification of all pairs of this type, I refer to [42], pages 154–170 for the proof of this part.

Finally, in the remaining case, there are are two intermediate subalgebras:

$$\mathfrak{g} \supsetneq \mathfrak{k}_1 \supsetneq \mathfrak{k}_2 \supsetneq \mathfrak{k},$$

so that Theorem 2.2.4 applies. $\qquad\square$

The last case in the proof of Theorem 2.2.5 is exactly the one that Sophus Lie does not handle in [42]. In geometric terms, it corresponds to an action of a local Lie group G on a three-dimensional manifold M which is very imprimitive in the following sense: M has an invariant foliation of curves, which in turn can be grouped together into two-dimensional surfaces that are also permuted by G.

2.7. Lie's Conjecture beyond Three Variables

We proved Lie's conjecture for transitive Lie algebras in $1, 2$, and 3 variables. The proof relies on the coincidence that an effective pair $(\mathfrak{g}, \mathfrak{k})$ of low codimension is likely either to be of the 'very imprimitive' sort (so that Theorem 2.2.4 applies), or to have a nice nilpotently acting subalgebra complementary to $\mathfrak{k}$, in which case we can apply Theorem 2.2.3. This is not true in higher codimensions, so our arguments fail there.

On the other hand, in order to find counter-examples, one needs tools to prove that some given transitive Lie algebra has no conjugate with coefficients in E. If one replaces E by the algebra of algebraic functions, i.e., the elements of $K[[\mathbf{x}]]$ satisfying some polynomial equation with coefficients from $K[\mathbf{x}]$, then the transcendence degree is such a tool. Indeed, consider the following transitive Lie algebra in two variables:

$$\mathfrak{g}_\Lambda := K\partial_x \oplus \bigoplus_{\lambda \in \Lambda} K\exp(\lambda x)\partial_y,$$

where Λ is a finite non-empty subset of K; this is type $(1,1)$ of Table 2 in Appendix A.

PROPOSITION 2.7.1. *The Lie algebra $\mathfrak{g}_\Lambda$ has a conjugate with algebraic coefficients if and only if the $\mathbb{Q}$-space*

$$W := \langle \lambda_1 - \lambda_2 \mid \lambda_1, \lambda_2 \in \Lambda \rangle_{\mathbb{Q}}$$

is at most 1-dimensional. In that case, $\mathfrak{g}_\Lambda$ even has a conjugate with polynomial coefficients.

PROOF. If $\dim_{\mathbb{Q}} W \leq 1$, then Λ is contained in a set of the form $\lambda_2 + \mathbb{N}\lambda_1$. Consider the coordinate change given by

$$u = \exp(\lambda_1 x) - 1, \quad v = y \exp(-\lambda_2 x).$$

It maps

$$\partial_x \mapsto \lambda_1(1 + u)\partial_u - \lambda_2 v \partial_v, \text{ and}$$

$$\exp((\lambda_2 + k\lambda_1)x)\partial_y \mapsto (1 + u)^k \partial_v \text{ for } k \in \mathbb{N}.$$

For the converse, suppose that $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ are such that $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are linearly independent over $\mathbb{Q}$. Consider any coordinate change

$$u = x + \ldots, \quad v = y + \ldots,$$

where the dots denote higher-order terms, and suppose that it maps $\mathfrak{g}_\Lambda$ to a transitive Lie algebra with coefficients that are algebraic over $K(u, v)$. It then follows that $f_i = \exp(\lambda_i - \lambda_3)x$ is algebraic over $K(u, v)$ for $i = 1, 2$. But this contradicts the fact that v, f_1, f_2 are algebraically independent over K. $\qquad\square$

Here we use the transcendence degree to decide whether certain given elements of $K[[\mathbf{x}]]$ together lie in some conjugate of the subalgebra of $K[[\mathbf{x}]]$ consisting of algebraic functions. For disproving Lie's conjecture, the availability of a similar tool distinguishing conjugates of E inside $K[[\mathbf{x}]]$ from wilder algebras, would be useful. Indeed, such a tool would also be of interest in its own right.

In view of Theorem 2.2.4, it is natural to search for counter-examples to Lie's conjecture on the other end of the spectrum of effective pairs: the primitive ones. Chapter 3 rules out some of these candidate counter-examples, by showing that they are subpairs of pairs that are known to have a polynomial realization.

CHAPTER 3

Primitive Lie Algebras

3.1. Introduction

In [**62**], Michel and Winternitz compute explicit polynomial transitive realizations of *classical-parabolic* primitive pairs, i.e., pairs $(\mathfrak{g}, \mathfrak{p})$ where $\mathfrak{g}$ is a finite-dimensional classical simple Lie algebra, and $\mathfrak{p}$ is a maximal parabolic subalgebra. For any such realization $\phi : \mathfrak{g} \to \hat{\mathfrak{D}}^{(n)}$, where $n = \mathrm{codim}_{\mathfrak{g}}\, \mathfrak{p}$, they wonder whether $\phi(\mathfrak{g})$ is maximal among the finite-dimensional subalgebras of $\hat{\mathfrak{D}}^{(n)}$. Michel and Winternitz work over the complex numbers, as does Dynkin in his classification of maximal subalgebras of the finite-dimensional simple Lie algebras. However, this classification does not change in replacing $\mathbb{C}$ by the arbitrary algebraically closed field K of characteristic 0 which is the ground field of all structures appearing in this chapter.

By the Realization Theorem, maximality of $\phi(\mathfrak{g})$ is equivalent to maximality of $(\mathfrak{g}, \mathfrak{p})$ among the finite-dimensional pairs of codimension n. This observation leads to the following order on pairs: we call a pair $(\mathfrak{g}_1, \mathfrak{k}_1)$ a *subpair* of a pair $(\mathfrak{g}_2, \mathfrak{k}_2)$ if there exists an injective morphism $\phi : (\mathfrak{g}_1, \mathfrak{k}_1) \to (\mathfrak{g}_2, \mathfrak{k}_2)$ in the sense of Section 1.1, and if moreover $\mathrm{codim}_{\mathfrak{g}_1} \mathfrak{k}_1 = \mathrm{codim}_{\mathfrak{g}_2} \mathfrak{k}_2$. Such a map ϕ is called an *inclusion* or *embedding* of $(\mathfrak{g}_1, \mathfrak{k}_1)$ into $(\mathfrak{g}_2, \mathfrak{k}_2)$. Note that in this case $\mathfrak{g}_2 = \phi(\mathfrak{g}_1) + \mathfrak{k}_2$. Conversely, if $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}_2$ such that $\mathfrak{g}_2 = \mathfrak{g}_1 + \mathfrak{k}_2$, then $(\mathfrak{g}_1, \mathfrak{g}_1 \cap \mathfrak{k}_2)$ is a subpair of $(\mathfrak{g}_2, \mathfrak{k}_2)$. If a finite-dimensional pair is not a subpair of any other finite-dimensional pair, then it is called *maximal*.

In terms of primitive pairs, Michel and Winternitz prove maximality of the classical-parabolic pairs $(\mathfrak{sl}_n, \mathfrak{p}_i)$ for all $n \geq 2$ and all $i = 1, \ldots, n-1$, where, as on page 7, the maximal parabolic subalgebras are labelled by the numbering of [**7**]. On the other hand, they find two families of inclusions among classical-parabolic pairs: $(\mathfrak{sp}_{2n}, \mathfrak{p}_1) \subset (\mathfrak{sl}_{2n}, \mathfrak{p}_1)$ for all $n \geq 1$, and $(\mathfrak{o}_{2n+1}, \mathfrak{p}_n) \subset (\mathfrak{o}_{2n+2}, \mathfrak{p}_{n+1})$ for all $n \geq 2$.

Michel and Winternitz conclude their article with the following question: is $(\mathfrak{o}_{10}, \mathfrak{p}_5)$ a subpair of $(\mathfrak{o}_{11}, \mathfrak{p}_3)$? These pairs have codimension 15, and this is the smallest codimension in which the dimension arguments that the authors use, do not suffice to establish non-existence of inclusions other than the ones they already found. For indeed, there are no other inclusions among classical-parabolic pairs, as we will see in Section 3.4; in particular, the answer to Michel and Winternitz's question is no. If we consider all *simple-parabolic pairs*, i.e., pairs $(\mathfrak{g}, \mathfrak{p})$ where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and $\mathfrak{p}$ is a maximal parabolic subalgebra, then we have precisely one more inclusion, namely $(G_2, \mathfrak{p}_1(G_2)) \subset (B_3, \mathfrak{p}_1(B_3))$. This is all proved in Section 3.4. Having thus found all inclusions among simple-parabolic pairs, we set out to find all inclusions of other finite-dimensional primitive pairs into simple-parabolic pairs in Section 3.5.

Solving Winternitz and Michel's problem is one motivation for this chapter; there are two more. First, I want to discuss Morozov's and Dynkin's classifications of finite-dimensional primitive pairs because of the important role they play in transitive differential geometry. Their classifications are the topic of Section 3.2. Second, as pointed out in Section 1.3, if $(\mathfrak{g}_1, \mathfrak{k}_1)$ is a subpair of a pair $(\mathfrak{g}_2, \mathfrak{k}_2)$, then any transitive realization of the latter pair restricts to a transitive realization of the former. In particular, if the latter pair is simple-parabolic so that it has a polynomial transitive realization by Example 2.4.1, then so has the former. This rules out many candidate counterexamples to Lie's conjecture of Chapter 2.

Ernest B. Vinberg informed me that many results of this chapter have been obtained earlier in the setting of compact Lie groups [**50**]. In particular, the problem of finding all inclusions among simple-parabolic pairs was already solved by Onishchik in the 1960s [**48**]. Also, our inclusion problem is related to that of finding factorizations of connected affine algebraic groups. Indeed, let G be such a group, let G_1, G_2 be closed subgroups, and suppose that $G = G_1 G_2$. Then $L(G) = L(G_1) + L(G_2)$, so that $(L(G_1), L(G_1 \cap G_2))$ is a subpair of $(L(G), L(G_2))$. Conversely, still under the assumption that G be connected, $L(G) = L(G_1) + L(G_2)$ implies that G_1 has a dense open orbit on G/G_2, from whence it follows that $G_1 G_2$ is an open dense subset of G. In general, however, it is a proper subset of G; see for example the geometric interpretation of Proposition 3.5.4. Under the condition that one of G_1, G_2 be parabolic, factorizations $G = G_1 G_2$ of simple algebraic G were classified by Kantor in characteristic 0 [**37**] and by Onishchik in arbitrary characteristic [**49**]. Liebeck, Saxl and Seitz classify factorizations under milder conditions in [**43**]. In contrast, the approach to the inclusion problem followed in this chapter is mostly Lie algebraic and very elementary.

3.2. Morozov's and Dynkin's Classifications

Recall the definition of a primitive pair $(\mathfrak{g}, \mathfrak{k})$ from Section 1.1. On one hand, $\mathfrak{k}$ must be maximal in $\mathfrak{g}$, but on the other hand, it must not contain any $\mathfrak{g}$-ideal. This makes it possible to classify finite-dimensional primitive pairs. We distinguish two cases: either $\mathfrak{g}$ is simple, or it is not.

THEOREM 3.2.1 (Morozov). *Let $(\mathfrak{g}, \mathfrak{k})$ be a finite-dimensional primitive pair, and assume that $\mathfrak{g}$ is not simple. Then one of the following holds.*

(1) *$g = \mathfrak{k} \ltimes \mathfrak{m}$, where $\mathfrak{k}$ acts faithfully and irreducibly on the Abelian ideal $\mathfrak{m}$.*
(2) *$g \cong \mathfrak{l} \oplus \mathfrak{l}$, where $\mathfrak{l}$ is simple and the two summands commute; and $\mathfrak{k}$ is the diagonal subalgebra $\{(l, l) \mid l \in \mathfrak{l}\}$.*

Conversely, in both cases the pair $(\mathfrak{g}, \mathfrak{k})$ is primitive.

For a proof of this theorem see [**21**]. Dynkin's classification of primitive pairs $(\mathfrak{g}, \mathfrak{k})$ with $\mathfrak{g}$ simple splits into the case where $\mathfrak{g}$ is classical and the case where it is exceptional. In the former case, i.e., if $\mathfrak{g}$ is of type A, B, C or D, then by the *standard module* of $\mathfrak{g}$ we mean the fundamental representation corresponding to the first node in the Dynkin diagram.

THEOREM 3.2.2 (Dynkin). *Let* $(\mathfrak{g}, \mathfrak{k})$ *be a finite-dimensional primitive pair, and assume that* $\mathfrak{g}$ *is classical simple. Let* V *be its standard module. There are two possibilities:*

$\mathfrak{k}$ **acts reducibly on** V**:** *in this case, one of the following holds.*

(1) $\mathfrak{k}$ *is a maximal parabolic subalgebra of* $\mathfrak{g}$. *These are of the form*

$$\mathfrak{p}(V') := \{g \in \mathfrak{g} \mid gV' \subseteq V'\}$$

with V' *a proper subspace of* V, *totally isotropic in case* $\mathfrak{g}$ *is orthogonal or symplectic. Moreover, if* $\mathfrak{g}$ *is of type* D_m, *then* $\dim U \neq m - 1$.

(2) $\mathfrak{g} = \mathfrak{o}(V)$, *and* $\mathfrak{k} = \mathfrak{o}(U) \oplus \mathfrak{o}(U^\perp)$ *for some non-degenerate* U, $0 \subsetneq U \subsetneq V$.

(3) $\mathfrak{g} = \mathfrak{sp}(V)$, *and* $\mathfrak{k} = \mathfrak{sp}(U) \oplus \mathfrak{sp}(U^\perp)$ *for some non-degenerate* U, $0 \subsetneq U \subsetneq V$.

$\mathfrak{k}$ **acts irreducibly on** V**:** *then there are two possibilities.*

$\mathfrak{k}$ **is not simple:** *then one of the following holds.*

(1) $\mathfrak{g} = \mathfrak{sl}(V)$, *where* $V \cong V_1 \otimes V_2$, *and* $\mathfrak{k} = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$. *Here* V_1, V_2 *have dimensions* ≥ 2.

(2) $\mathfrak{g} = \mathfrak{sp}(V)$, *where* $V \cong V_1 \otimes V_2$, *and* $\mathfrak{k} = \mathfrak{sp}(V_1) \oplus \mathfrak{o}(V_2)$. *Here* $\dim V_1 \geq 2, \dim V_2 \geq 3$ *but either* $\dim V_2 \neq 4$ *or* $(\dim V_1, \dim V_2)$ *equals* $(2, 4)$. *Moreover,* V_1 *is equipped with a non-degenerate skew bilinear form, and* V_2 *with a non-degenerate symmetric bilinear form, such that the skew form on* V *is the product of the two.*

(3) $\mathfrak{g} = \mathfrak{o}(V)$, *where* $V \cong V_1 \otimes V_2$, *and* $\mathfrak{k} = \mathfrak{o}(V_1) \oplus \mathfrak{o}(V_2)$. *Here* $\dim V_1, \dim V_2 \geq 3$ *but* $\neq 4$. *Moreover, each* V_i *is equipped with a non-degenerate symmetric bilinear form, such that the symmetric form on* V *is the product of the two.*

(4) $\mathfrak{g} = \mathfrak{o}(V)$, *where* $V \cong V_1 \otimes V_2$, *and* $\mathfrak{k} = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$. *Here* $\dim V_1, \dim V_2 \geq 2$. *Moreover, each* V_i *is equipped with a non-degenerate skew bilinear form, such that the symmetric form on* V *is the product of the two.*

$\mathfrak{k}$ **is simple:** *then one of the following holds.*

(1) $\mathfrak{g} = \mathfrak{sl}(V)$, *and* $\mathfrak{k} = \mathfrak{o}(V)$ *for some non-degenerate symmetric bilinear form on* V.

(2) $\mathfrak{g} = \mathfrak{sl}(V)$, *and* $\mathfrak{k} = \mathfrak{sp}(V)$ *for some non-degenerate skew bilinear form on* V.

(3) $\mathfrak{g} = \mathfrak{sl}(V)$, *and* $\mathfrak{k}$ *leaves invariant no bilinear form on* V.

(4) $\mathfrak{g} = \mathfrak{o}(V)$, *and* $\mathfrak{k}$ *leaves invariant the symmetric bilinear form on* V *defining* $\mathfrak{g}$.

(5) $\mathfrak{g} = \mathfrak{sp}(V)$, *and* $\mathfrak{k}$ *leaves invariant the skew bilinear form on* V *defining* $\mathfrak{g}$.

Conversely, still assuming that $\mathfrak{g}$ *is classical simple, if* $(\mathfrak{g}, \mathfrak{k})$ *appears in the above list, then it is, as a rule, a primitive pair. Only in the last three cases, there are exceptions to this rule, and they are listed in Table 1 of* [**18**].

Dynkin proves this in [**18**]. In [**19**], he also classifies the maximal subalgebras of the exceptional simple Lie algebras.

REMARK 3.2.3. A note concerning the exception $\dim V_2 = 4$ in the second half of Dynkin's theorem: from $\mathfrak{o}_4 \cong \mathfrak{sp}_2 \oplus \mathfrak{sp}_2$ we find that, for $n \geq 1$,

$$\mathfrak{sp}_{2n} \oplus \mathfrak{o}_4 = (\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2) \oplus \mathfrak{sp}_2$$
$$\subseteq \mathfrak{o}_{4n} \oplus \mathfrak{sp}_2$$
$$\subsetneq \mathfrak{o}_{8n},$$

where the first inclusion is strict if and only if $n > 1$. This explains why $(\mathfrak{sp}_{2n} \oplus \mathfrak{o}_4)$ is not a maximal subalgebra of $\mathfrak{o}_{8n}$ unless $n = 1$. Similarly, for $n \geq 3$, we have:

$$\mathfrak{o}_n \oplus \mathfrak{o}_4 = (\mathfrak{o}_n \oplus \mathfrak{sp}_2) \oplus \mathfrak{sp}_2$$
$$\subsetneq \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$$
$$\subsetneq \mathfrak{o}_{4n},$$

which shows that $\mathfrak{o}_n \oplus \mathfrak{o}_4$ is not a maximal subalgebra of $\mathfrak{o}_{4n}$.

Morozov's and Dynkin's results imply that the number of isomorphism classes of primitive pairs of a fixed codimension is finite. To get a good insight into the inclusion problem, I wrote a program in the computer algebra program LiE that determines all classes of a given codimension. Extensive use of this program taught me which dimension arguments to use in what follows. Moreover, in order to discard candidate inclusions in 'low' codimensions, I sometimes use this program, but, for the sake of brevity, I do not always include its complete output.

3.3. Maximal Parabolic Subalgebras

The maximal parabolic subalgebras of a simple Lie algebra $\mathfrak{g}$ are parameterized by the vertices of the Dynkin diagram of $\mathfrak{g}$; see page 7. Their geometric interpretations are as follows. For $\mathbf{m} \in \mathbb{N}^l$, we denote by $V_{\mathbf{m}}$ the irreducible $\mathfrak{g}$-module with highest weight $\mathbf{m}$ with respect to the basis of fundamental weights. In $V_{\mathbf{e}_i}$, the highest weight line $Kv_{\mathbf{e}_i}$ is stabilized by $\mathfrak{p}_i$; as the latter Lie algebra is maximal in $\mathfrak{g}$, it can be characterized as the stabilizer of $Kv_{\mathbf{e}_i}$.

Maximal parabolic subalgebras of classical simple Lie algebras have yet another interpretation, as indicated in Theorem 3.2.2. By $\mathfrak{cl}(V)$, we denote any of the classical Lie algebras with standard module V. By $\mathfrak{p}(V')$, we denote the maximal parabolic subalgebra of $\mathfrak{cl}(V)$ consisting of maps leaving V' invariant. If $\mathfrak{cl}(V)$ is orthogonal or symplectic, then V is understood to be endowed with a non-degenerate symmetric or skew bilinear form, respectively, and V' is totally isotropic with respect to this form. In addition, if $\mathfrak{cl}(V)$ is of type D_m, then $\dim V' \neq m - 1$, as any $(m-1)$-dimensional totally isotropic subspace V' of V is contained in precisely two m-dimensional isotropic subspaces, and both are invariant under any element of $\mathfrak{o}(V)$ leaving V' invariant. Note that $\mathfrak{p}(V')$ is always conjugate to $\mathfrak{p}_{\dim V'}$ by an automorphism of $\mathfrak{cl}(V)$ (possibly outer if $\mathfrak{cl}(V)$ is of type D_n).

We often need the codimensions of maximal parabolic subalgebras in simple ones; the numbers in the following lemma are easily obtained from the explicit description of the classical Lie algebras in [35], and by calculations with LiE [57] for the exceptional Lie algebras.

LEMMA 3.3.1. *Let n be the semisimple rank of $\mathfrak{cl}(V)$, set $n_1 := \dim V' > 0$, and write $n = n_1 + n_2$. Then the codimension of $\mathfrak{p}(V')$ in $\mathfrak{cl}(V)$ equals*

$$n_1(n_2 + 1) \text{ if } \mathfrak{cl}(V) \text{ is of type } A_n,$$

$$2n_1 n_2 + \binom{n_1 + 1}{2} \text{ if } \mathfrak{cl}(V) \text{ is of type } B_n \text{ or } C_n, \text{ and}$$

$$2n_1 n_2 + \binom{n_1}{2} \text{ if } \mathfrak{cl}(V) \text{ is of type } D_n.$$

Furthermore, the codimensions of maximal parabolic subalgebras of the exceptional simple Lie algebras are as follows:

$$5, 5 \text{ for type } G_2,$$
$$15, 20, 20, 15 \text{ for type } F_4,$$
$$16, 21, 25, 29, 25, 16 \text{ for type } E_6,$$
$$33, 42, 47, 53, 50, 42, 27 \text{ for type } E_7, \text{ and}$$
$$78, 92, 98, 106, 104, 97, 83, 57 \text{ for type } E_8.$$

Consider a classical-parabolic pair $(\mathfrak{cl}(U), \mathfrak{p}(U'))$. The following lemma describes the structure of U as a $\mathfrak{p}(U')$-module; it is easily deduced from the explicit description of the classical Lie algebras in [**35**].

LEMMA 3.3.2. *Let $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ be a classical-parabolic pair. If it is not of type $(B_n, \mathfrak{p}_n)$, then $0, U'$, and U are the only $\mathfrak{p}(U')$-invariant subspaces of U. If it is, then $0, U', (U')^\perp \supset U'$, and U are the only $\mathfrak{p}(U')$-invariant subspaces of U.*

We call two $\mathfrak{g}$-modules V and W *equivalent* if there is an automorphism of $\mathfrak{g}$ carrying one over into the other. Then $V_{\mathbf{m}}$ is equivalent to $V_{\mathbf{n}}$ if and only if there is an automorphism of the Dynkin diagram of $\mathfrak{g}$ mapping $\mathbf{m}$ to $\mathbf{n}$.

Using the following lemma, we can often deduce that a given $\mathfrak{cl}(U)$-module cannot have irreducible submodules other than U, U^*, and trivial ones.

LEMMA 3.3.3. *Let $\mathfrak{cl}(U)$ be classical simple. Then its non-trivial irreducible modules of smallest dimension are U and U^*, unless $\mathfrak{cl}(U)$ is $\mathfrak{o}_5$, in which case it is isomorphic to $\mathfrak{sp}_4$. Apart from U and U^* the smallest non-trivial irreducible modules are*

(1) *the exterior square $V_{\mathbf{e}_2} = \Lambda^2(U)$, if $\mathfrak{cl}(U)$ is of type A, B_n with $n \geq 7$ or D_n with $n \geq 8$; it has dimension $\binom{\dim U}{2}$,*
(2) *the spin module $V_{\mathbf{e}_n}$, for type B_n with $n < 7$, of dimension 2^n,*
(3) *the spin modules $V_{\mathbf{e}_{n-1}}$ and $V_{\mathbf{e}_n}$ for type D_n with $n < 8$, of dimension 2^{n-1},*
(4) *$V_{\mathbf{e}_2}$, which is $\Lambda^2(U)$ minus a trivial one-dimension submodule, for type C_n with $n \geq 4$; it has dimension $\binom{\dim V}{2} - 1$, and*
(5) *$V_{\mathbf{e}_2}$ and $V_{\mathbf{e}_3}$ for type C_3, both of dimension 14.*

For exceptional simple Lie algebras, the smallest faithful modules are the following:

(1) *for E_6: $V_{\mathbf{e}_1}$ and $V_{\mathbf{e}_6}$, both of dimension 27,*
(2) *for E_7: $V_{\mathbf{e}_7}$, of dimension 56,*
(3) *for E_8: $V_{\mathbf{e}_8}$, of dimension 248,*
(4) *for F_4: $V_{\mathbf{e}_4}$, of dimension 26, and*
(5) *for G_2: $V_{\mathbf{e}_1}$, of dimension 7.*

PROOF. Weyl's dimension formula ([**35**], Chapter VIII) implies $\dim V_{\mathbf{m}} > \dim V_{\mathbf{n}}$ if $\mathbf{m} \neq \mathbf{n}$ and $m_i \geq n_i$ for all i. Hence, any non-trivial module other than $U \cong V_{\mathbf{e}_1}$ has dimension at least the minimum of the dimensions of $V_{2\mathbf{e}_1}$ and $V_{\mathbf{e}_i}$ for $i \neq 1$. Now, the result follows from the tables in [**9**]. $\qquad\square$

3.4. Inclusions among Simple-Parabolic Pairs

The following propositions describe some inclusions among simple-parabolic pairs.

PROPOSITION 3.4.1. *The pair* $(\mathfrak{sp}_{2n}, \mathfrak{p}_1(\mathfrak{sp}_{2n}))$ *is a subpair of* $(\mathfrak{sl}_{2n}, \mathfrak{p}_1(\mathfrak{sl}_{2n}))$, *for any integer* $n \geq 2$.

This proposition appears as Lemma 6.1 in [**62**].

PROOF. Let V be a $2n$-dimensional vector space equipped with a non-degenerate skew bilinear form, and let $v \in V$ be non-zero. The embedding $\phi : \mathfrak{sp}(V) \to \mathfrak{sl}(V)$ is the natural one, and it is clear that the pre-image of the stabilizer in $\mathfrak{sl}(V)$ of Kv under ϕ is the stabilizer in $\mathfrak{sp}(V)$ of Kv. Furthermore, both codimensions equal $n - 1$ by Lemma 3.3.1. $\qquad\square$

PROPOSITION 3.4.2. *Let* n *be an integer* ≥ 2. *Let* V *be a* $(2n + 2)$-*dimensional vector space equipped with a symmetric bilinear form,* $V' \subseteq V$ *an* $(n + 1)$-*dimensional totally isotropic subspace, and* U *a* $(2n + 1)$-*dimensional non-degenerate subspace of* V. *Then* $U' := U \cap V'$ *is an* n-*dimensional totally isotropic subspace of* U, *and the pair* $(\mathfrak{o}(U), \mathfrak{p}(U'))$ *is a subpair of* $(\mathfrak{o}(V), \mathfrak{p}(V'))$.

This proposition is Lemma 5.2 of [**62**].

PROOF. First, $U \cap V'$ has dimension at least $2n + 1 + n + 1 - (2n + 2) = n$. As it is totally isotropic, it has dimension $\leq \lfloor \dim(U)/2 \rfloor = n$. This proves the first statement.

Define $\phi : \mathfrak{o}(U) \to \mathfrak{o}(V)$ by $\phi(A)|_U = A$ and $\phi(A)|_{U^\perp} = 0$. Now if $A \in \mathfrak{p}(U')$, then $\phi(A)$ leaves U' invariant, and also the two $(n + 1)$-dimensional totally isotropic subspaces containing U'. One of these is V', so that $\phi(A) \in \mathfrak{p}(V')$. Conversely, if $\phi(A)$ leaves V' invariant, then it leaves $V' \cap U = U'$ invariant. Hence

$$\phi^{-1}(\mathfrak{p}(V')) = \mathfrak{p}(U').$$

Finally, the codimensions of both pairs equal $\binom{n+1}{2}$. $\qquad\square$

PROPOSITION 3.4.3. *The pair* $(G_2, \mathfrak{p}_1(G_2))$ *is a subpair of* $(\mathfrak{o}_7, \mathfrak{p}_1(\mathfrak{o}_7))$.

PROOF. The 7-dimensional irreducible G_2-module $V_{(1,0)}$ has a non-degenerate symmetric G_2-invariant bilinear form; this defines the embedding $\phi : G_2 \to \mathfrak{o}(7)$. The parabolic subalgebra $\mathfrak{p}_1$ of G_2 is the stabilizer of the line Kv spanned by the highest weight vector $v \in V_{(1,0)}$. This vector is isotropic, and we find $\mathfrak{p}_1(G_2) = \phi^{-1}(\mathfrak{p}(Kv))$. Finally, both pairs have codimension 5. $\qquad\square$

A major part of this section is devoted to the proof of the following theorem.

THEOREM 3.4.4. *Propositions 3.4.1, 3.4.2 and 3.4.3 describe all inclusions among primitive simple-parabolic pairs.*

Roughly, the proof runs as follows. First we restrict our attention to inclusions $\phi : (\mathfrak{cl}(U), \mathfrak{p}(U')) \to (\mathfrak{cl}(V), \mathfrak{p}(V'))$ among classical-parabolic pairs. Consider V as a $\mathfrak{cl}(U)$-module through ϕ. For dimension reasons, it will turn out that V splits into modules that are either trivial or equivalent to U. Using this, we prove that both $\dim V' \geq \dim U'$ and $\mathrm{codim}_V V' \geq \mathrm{codim}_U U'$. By dimension arguments, this permits us to solve the case where $\mathfrak{cl}(V) = \mathfrak{sl}(V)$, and to show that if $\mathfrak{cl}(V)$ is orthogonal or symplectic, then so is $\mathfrak{cl}(U)$. Hence, only the inclusions among orthogonal pairs and those among symplectic pairs remain to be treated. For these, we show that $V \cong U \oplus T$ as a $\mathfrak{cl}(U)$-module, where T is trivial, and that V' splits accordingly into $U' \oplus T'$. Again, a dimension argument shows the non-existence of inclusions other than the ones that we have already found. Finally, we deal with the exceptional-parabolic pairs.

Before carrying on with the details, I note that Gerhard Post proves maximality of certain *graded* primitive Lie algebras in an entirely different way in [**51**]. Within the realm of simple-parabolic pairs, these are the pairs $(\mathfrak{g}, \mathfrak{p}_i)$ where $\mathfrak{g}$ is simple and i corresponds to a simple root whose coefficient in the highest root is 1; representation theory associated to these pairs is the subject of Chapter 5. More generally, he derives sufficient conditions for a given finite-dimensional transitive graded Lie algebra to be contained in a unique maximal finite-dimensional transitive graded Lie algebra.

Inclusions among Classical-Parabolic Pairs. If ϕ is an embedding of the pair $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ into $(\mathfrak{cl}(V), \mathfrak{p}(V'))$, then V can be considered as a $\mathfrak{cl}(U)$-module through ϕ. We want to prove that non-standard $\mathfrak{cl}(U)$-modules are too large to fit into V.

LEMMA 3.4.5. *For any classical-parabolic pair $(\mathfrak{cl}(V), \mathfrak{p}(V'))$ we have*

$$\dim V \leq \mathrm{codim}_{\mathfrak{cl}(V)} \mathfrak{p}(V') + 2.$$

PROOF. Let n be the semisimple rank of $\mathfrak{cl}(V)$, $n_1 := \dim V'$. From Lemma 3.3.1, we find that $\mathrm{codim}_{\mathfrak{cl}(V)} \mathfrak{p}(V')$ is minimal for $n_1 = 1$. In this case, it equals $n, 2n-1, 2n-1$, or $2n-2$ according to whether $\mathfrak{cl}(V)$ is of type A_n, B_n, C_n or D_n. The lemma holds in each case. $\square$

LEMMA 3.4.6. *Consider a classical-parabolic pair $(\mathfrak{cl}(U), \mathfrak{p}(U'))$, and let V be an irreducible $\mathfrak{cl}(U)$-module which is neither trivial nor equivalent to U. Then*

$$\dim V > \mathrm{codim}_{\mathfrak{cl}(U)} \mathfrak{p}(U') + 2,$$

unless $(\mathfrak{cl}(U), \mathfrak{p}(U'), V)$ is one of the following triples:

(6)
$$(\mathfrak{sl}_2, \mathfrak{p}_1, V_{(2)}), \quad (\mathfrak{sl}_4, \mathfrak{p}_2, V_{(0,1,0)}), \quad (\mathfrak{o}_5, \mathfrak{p}_1, V_{(0,1)}),$$
$$(\mathfrak{o}_5, \mathfrak{p}_2, V_{(0,1)}), \quad (\mathfrak{o}_7, \mathfrak{p}_2, V_{(0,0,1)}), \quad (\mathfrak{o}_7, \mathfrak{p}_3, V_{(0,0,1)}), \; or$$
$$(\mathfrak{o}_{10}, \mathfrak{p}_3, V_{\mathbf{e}_6}).$$

PROOF. Let n be the semisimple rank of $\mathfrak{cl}(U)$, set $n_1 := \dim U'$, and write $n = n_1 + n_2$. Now distinguish between the four possible types of $\mathfrak{cl}(U)$.

$\mathfrak{cl}(U)$ **of type** A_n**:** By Lemma 3.3.3, we have

$$\dim V \geq \binom{n+1}{2} = n_1(n_2 + 1) + \binom{n_1}{2} + \binom{n_2 + 1}{2},$$

which is larger than $n_1(n_2 + 1) + 2 = \mathrm{codim}_{\mathfrak{cl}(U)} \mathfrak{p}(U') + 2$ unless we are in the first two exceptions of the lemma.

$\mathfrak{cl}(U)$ **of type** B_n**:** If $n \geq 7$, then by Lemma 3.3.3

$$\dim V \geq \binom{2n+1}{2} = 4n_1 n_2 + \binom{2n_1+1}{2} + \binom{2n_2+1}{2},$$

which is larger than $2n_1 n_2 + \binom{n_1}{2} + 2$. The cases where $n = 2, \ldots, 7$ yield four exceptions, listed in the lemma.

$\mathfrak{cl}(U)$ **of type** C_n**:** Now by Lemma 3.3.3

$$\dim V \geq \binom{2n}{2} - 1 = 4n_1 n_2 + \binom{2n_1}{2} + \binom{2n_2}{2} - 1,$$

which is larger than $2n_1 n_2 + \binom{n_1+1}{2} + 2$ if $n \geq 3$, which we may assume as we have already dealt with B_2.

$\mathfrak{cl}(U)$ **of type** D_n**:** If $n \geq 8$, then by Lemma 3.3.3

$$\dim V \geq \binom{2n}{2} = 4n_1 n_2 + \binom{2n_1}{2} + \binom{2n_2}{2};$$

this is larger than $2n_1 n_2 + \binom{n_1}{2} + 2$. The cases where $n = 4, \ldots, 8$ yield the last exception in the lemma.

$\square$

Let us get rid of the exceptions (6).

LEMMA 3.4.7. *The only proper inclusions of a pair* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *from the list* (6) *into another classical-parabolic pair are:*

$$(\mathfrak{o}_5, \mathfrak{p}_2) \to (\mathfrak{o}_6, \mathfrak{p}_3) \ and \ (\mathfrak{o}_7, \mathfrak{p}_3) \to (\mathfrak{o}_8, \mathfrak{p}_4);$$

both are described by Proposition 3.4.2.

PROOF.

(1) The pair $(\mathfrak{sl}_2, \mathfrak{p}_1)$ is the only classical-parabolic pair of codimension 1, by Lemma 3.3.1.

(2) Apart from the pair $(\mathfrak{sl}_4, \mathfrak{p}_2)$, there is only one other classical-parabolic pair of codimension 4, namely $(\mathfrak{sl}_5, \mathfrak{p}_1)$. Consider an inclusion $\phi : \mathfrak{sl}_4 \to \mathfrak{sl}_5$. The standard module V of the latter splits into a module U equivalent to the standard module for $\mathfrak{sl}_4$, and a trivial one. Hence, $\mathfrak{p}_2 \subseteq \mathfrak{sl}_4$ does not leave invariant a one-dimensional subspace of V other than the trivial $\mathfrak{sl}_4$-module, i.e., $\phi(\mathfrak{p}_2)$ cannot be a subalgebra of $\mathfrak{p}_1 \subseteq \mathfrak{sl}_5$.

(3) Apart from the pair $(\mathfrak{o}_5, \mathfrak{p}_1)$, there are only two classical-parabolic pairs of codimension 3: $(\mathfrak{sl}_4, \mathfrak{p}_1)$, and $(\mathfrak{o}_5, \mathfrak{p}_2)$. The latter does not contain $(\mathfrak{o}_5, \mathfrak{p}_1)$, as the two parabolic subalgebras are not conjugate. Any embedding $\phi : \mathfrak{o}_5 \to \mathfrak{sl}_4$ is given by the 4-dimensional irreducible module $V_{(0,1)}$ for $\mathfrak{o}_5$. Suppose that $\mathfrak{p}_1 \subseteq \mathfrak{o}_5$ leaves invariant a line Kv in $V_{(0,1)}$. Then Kv is invariant under the Borel subalgebra of $\mathfrak{o}_5$ contained in $\mathfrak{p}_1$, so that Kv is the highest weight line, of weight $(0,1)$. But the stabilizer of this line is $\mathfrak{p}_2 \subseteq \mathfrak{o}_5$, which is not conjugate to $\mathfrak{p}_1$. Hence, $(\mathfrak{o}_5, \mathfrak{p}_1)$ is not a subpair of $(\mathfrak{sl}_4, \mathfrak{p}_1)$.

(4) The pair $(\mathfrak{o}_5, \mathfrak{p}_2)$ is not contained in $(\mathfrak{o}_5, \mathfrak{p}_1)$, but it is contained in the remaining classical-parabolic pair of codimension 3, namely $(\mathfrak{sl}_4, \mathfrak{p}_1) \cong (\mathfrak{o}_6, \mathfrak{p}_3)$.

(5) Apart from $(\mathfrak{o}_7, \mathfrak{p}_2)$, there are four classical-parabolic pairs of codimension 7: $(\mathfrak{sl}_8, \mathfrak{p}_1)$, $(\mathfrak{o}_9, \mathfrak{p}_1)$, $(\mathfrak{sp}_6, \mathfrak{p}_2)$, and $(\mathfrak{sp}_8, \mathfrak{p}_1)$. The parabolic subalgebra $\mathfrak{p}_2 \subseteq \mathfrak{o}_7$ does not leave a line invariant in the standard module $V_{\mathbf{e}_1}$ for $\mathfrak{o}_7$, nor in the spin module $V_{\mathbf{e}_3}$ for the latter. This shows that $(\mathfrak{o}_7, \mathfrak{p}_2)$ is not a subpair of the first two pairs, nor of the last pair. Also, $\mathfrak{o}_7$ does not fit into $\mathfrak{sp}_6$, which rules out the third one.

(6) The pair $(\mathfrak{o}_7, \mathfrak{p}_3)$ is only contained in the pair $(\mathfrak{o}_8, \mathfrak{p}_4)$. Note that the latter pair is isomorphic to $(\mathfrak{o}_8, \mathfrak{p}_1)$ and to $(\mathfrak{o}_8, \mathfrak{p}_3)$.

(7) Apart from $(\mathfrak{o}_{10}, \mathfrak{p}_3)$, the only classical-parabolic pairs of codimension 15 are

$$(\mathfrak{sl}_{16}, \mathfrak{p}_1), \quad (\mathfrak{sl}_8, \mathfrak{p}_3), \quad (\mathfrak{o}_{11}, \mathfrak{p}_2),$$
$$(\mathfrak{o}_{11}, \mathfrak{p}_5), \quad (\mathfrak{o}_{17}, \mathfrak{p}_1), \quad (\mathfrak{sp}_{10}, \mathfrak{p}_2),$$
$$(\mathfrak{sp}_{10}, \mathfrak{p}_5), (\mathfrak{sp}_{16}, \mathfrak{p}_1), \text{ and } (\mathfrak{o}_{12}, \mathfrak{p}_6).$$

Arguments like the ones in the previous cases show that $(\mathfrak{o}_{10}, \mathfrak{p}_3)$ is contained in neither of these.

$\square$

The use of the rather far-fetched Lemmas 3.4.5 and 3.4.6 becomes clear in the following lemma.

LEMMA 3.4.8. *Suppose that* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is a proper subpair of* $(\mathfrak{cl}(V), \mathfrak{cl}(V'))$. *Then we can write*

$$V = T \oplus \bigoplus_{i=1}^{k} U_i$$

as $\mathfrak{cl}(U)$*-modules, for some* $k \geq 1$, *where each* U_i *is equivalent to* U, *and* T *is a trivial module.*

PROOF. If $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is one of the exceptions in Lemma 3.4.6, then the statement holds by Lemma 3.4.7. Now assume that it is not one of those exceptions. The codimensions of both pairs are equal, say c. According to Lemma 3.4.5, $\dim V \leq c + 2$. By Lemma 3.4.6, any non-trivial irreducible $\mathfrak{cl}(U)$-module that is not equivalent to the standard one, has dimension $> c + 2$, hence does not fit into V. In order that the homomorphism $\mathfrak{cl}(U) \to \mathfrak{cl}(V)$ be nontrivial, we need $k \geq 1$. $\square$

This lemma will yield inequalities among the parameters of two classical-parabolic pairs, one of which is a subpair of the other; these inequalities will rule out many possible inclusions. First, however, it is convenient to rule out D_4, because—apart from A_n—it is the only classical simple Lie algebra having modules that are equivalent to, but not isomorphic to, its standard module.

LEMMA 3.4.9. *None of the pairs* $(\mathfrak{o}_8, \mathfrak{p}_i)$ *for* $i = 1, \ldots, 4$ *is a proper subpair of another classical-parabolic pair.*

PROOF. The codimension of $(\mathfrak{o}_8, \mathfrak{p}_i)$ equals 6 for $i = 1, 3, 4$ and 9 for $i = 2$. The other classical-parabolic pairs of codimension 6 are $(\mathfrak{sl}_5, \mathfrak{p}_2)$, $(\mathfrak{sl}_7, \mathfrak{p}_1)$, $(\mathfrak{o}_7, \mathfrak{p}_3)$, and $(\mathfrak{sp}_6, \mathfrak{p}_3)$. For none of these, the first component contains $\mathfrak{o}_8$, which proves the lemma for $i = 1, 3$, and 4.

The other classical-parabolic pairs of codimension 9 are $(\mathfrak{sl}_6, \mathfrak{p}_3)$, $(\mathfrak{sl}_{10}, \mathfrak{p}_1)$, $(\mathfrak{o}_{11}, \mathfrak{p}_1)$, and $(\mathfrak{sp}_{10}, \mathfrak{p}_1)$. If $(\mathfrak{o}_8, \mathfrak{p}_2)$ is to be a subpair of one of these, then $\mathfrak{o}_8$ must have a 10- or 11-dimensional module V on which $\mathfrak{p}_2$ is the stabilizer of a one-dimensional subspace V'. Now V is of the form $V_{\mathbf{e}_i} \oplus T$, where $i = 1, 3$, or 4, and T is a trivial module of dimension 2 or 3. Let π be the projection of V onto $V_{\mathbf{e}_i}$ along T. Then $\pi(V')$ must be non-zero, lest all of $\mathfrak{o}_8$ leave V' invariant. Hence $\pi(V')$ is one-dimensional, and invariant under $\mathfrak{p}_2$. In particular, it is invariant under the Borel subalgebra contained in $\mathfrak{p}_2$, hence $\pi(V')$ is the highest weight line, of weight $\mathbf{e}_i$. But the stabilizer of this line is $\mathfrak{p}_i$, which does not contain $\mathfrak{p}_2$. $\qquad\square$

LEMMA 3.4.10. *Suppose that $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is a subpair of $(\mathfrak{cl}(V), \mathfrak{p}(V'))$. If $\mathfrak{cl}(U)$ is of type B, C or D, and $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is not of type $(B_n, \mathfrak{p}_n)$, then we have*

(7)
$$\dim V' \geq \dim U', \ and$$
$$\operatorname{codim}_V V' \geq \operatorname{codim}_U U'.$$

If $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is of type $(B_n, \mathfrak{p}_n)$, then we have

$$\dim V' \geq \dim U', \ and$$
$$\operatorname{codim}_V V' \geq \operatorname{codim}_U U' - 1.$$

If $\mathfrak{cl}(U)$ is of type A, then at least one of (7) and the following holds:

$$\dim V' \geq \operatorname{codim}_U U', \ and$$
$$\operatorname{codim}_V V' \geq \dim U'.$$

PROOF. According to Lemma 3.4.8,

$$V = T \oplus \bigoplus_{i=1}^{k} U_i$$

as a $\mathfrak{cl}(U)$-module; here T is trivial and each U_i is equivalent to U.

First assume that $\mathfrak{cl}(U)$ is not $\mathfrak{sl}(U)$; then each U_i is isomorphic to U as Lemma 3.4.9 rules out $\mathfrak{o}_8$. For all $i = 1, \ldots, k$, let U_i' be the image of U' under an isomorphism $U \to U_i$ of $\mathfrak{cl}(U)$-modules, and let π_i be the projection of V onto U_i along $T \oplus \bigoplus_{j \neq i} U_j$. Then each $\pi_i(V')$ is invariant under $\mathfrak{p}(U')$. One of these must be non-zero, lest V' be invariant under all of $\mathfrak{cl}(U)$. From Lemma 3.3.2 it follows that $\pi_i(V') \supseteq U_i'$ for some i, which proves the first inequality. Furthermore, each $V' \cap U_i$ is invariant under $\mathfrak{p}(U')$. It cannot be all of U_i for all i, for then V' would again be invariant under $\mathfrak{cl}(U)$. Hence, for at least one i, it equals U_i', or $(U_i')^{\perp} \cap U_i$ if $\mathfrak{cl}(U)$ is of type B_n and U' is n-dimensional (see Lemma 3.3.2). This proves the second inequality.

If $\mathfrak{cl}(U)$ is $\mathfrak{sl}(U)$, then it is possible that some of the U_i are isomorphic to U^*, for which a similar reasoning as the above leads to the second pair of inequalities. $\qquad\square$

Now suppose, for instance, that $(\mathfrak{sl}(U), \mathfrak{p}(U'))$ is a subpair of $(\mathfrak{o}(V), \mathfrak{p}(V'))$, where $\dim V \geq 5$ and odd. Let $n := \dim(U) - 1$ be the semisimple rank of $\mathfrak{sl}(U)$, and set $n_1 := \dim U'$. Similarly, let $m := (\dim(V) - 1)/2 \geq 2$ be the rank of $\mathfrak{o}(V)$, and set $m_1 := \dim V'$. We clearly have $m \geq n$, and by the above lemma $m_1 \geq n_1$, possibly
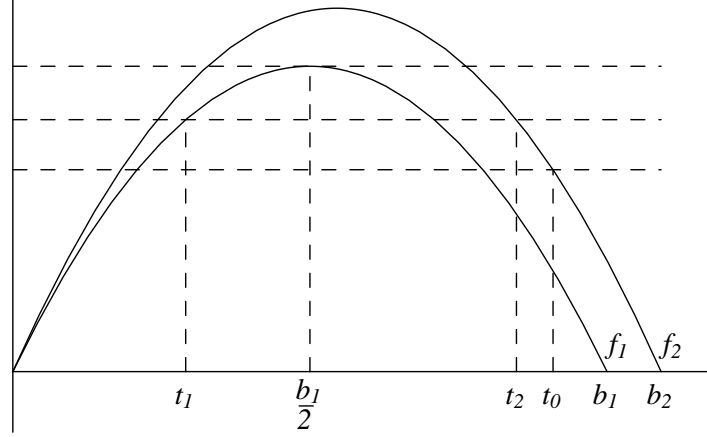
FIGURE 1. Illustration of Lemma 3.4.11.

after replacing $(\mathfrak{sl}(U), \mathfrak{p}(U'))$ by the isomorphic pair $(\mathfrak{sl}(U^*), \mathfrak{p}((U')^0))$, where $(U')^0$ is the annihilator of U' in U^*. On the other hand, we have

$$n_1(n + 1 - n_1) = 2m_1(m - m_1) + \binom{m_1 + 1}{2} = \frac{3}{2}m_1(\frac{4}{3}m + \frac{1}{3} - m_1)$$

by Lemma 3.3.1. These facts lead to a contradiction by the following lemma.

LEMMA 3.4.11. *For $i = 1, 2$, let $a_i, b_i \in \mathbb{R}_+$, and suppose that $a_1 b_1 \leq a_2 b_2$, $b_1 \leq b_2$, and at least one of these inequalities is strict. Define the quadratic real polynomials $f_i(t) := a_i t(b_i - t)$, $i = 1, 2$. Let $t_0 \in [0, b_2]$, and suppose that there exist t_1, t_2 with $0 < t_1 \leq t_2 \leq t_0$ and $f_1(t_1) = f_2(t_2)$. Then $f_1(b_1/2) \geq f_2(t_0)$.*

This lemma is best illustrated by Figure 1, rather than proved rigorously: from the conditions it follows that the positive part of the graph of f_2 lies entirely above that of f_1, so that $f_1(t_1) = f_2(t_2)$ and $t_0 \geq t_2 \geq t_1$ imply $f_2(t_2) \geq f_2(t_0)$, and, *a fortiori*, the maximum of f_1, which is attained in $b_1/2$, is at least $f_2(t_0)$.

Now take $a_1 = 1$, $b_1 = n + 1$, $a_2 = \frac{3}{2}$, and $b_2 = \frac{4}{3}m + \frac{1}{3}$. As $m \geq 2$, we have $b_1 = n + 1 \leq m + 1 \leq \frac{4}{3}m + \frac{1}{3} = b_2$; from $a_1 < a_2$ we find that also $a_1 b_1 < a_2 b_2$. Hence, taking $t_0 = m$, the conditions of the lemma are satisfied. We conclude that

$$\frac{1}{4}(n + 1)^2 \geq \frac{1}{2}m(m + 1),$$

so that

$$n \geq \sqrt{2m(m + 1)} - 1 > m,$$

a contradiction with $m \geq n$. Summarizing, the assumption that $(\mathfrak{sl}(U), \mathfrak{p}(U'))$ be a subpair of $(\mathfrak{o}(V), \mathfrak{p}(V'))$ leads, by Lemma 3.4.10, to inequalities among the parameters n, n_1, m, and m_1 that are contradictory by Lemma 3.4.11.

Consider another exemplary situation where the above lemmas serve us well in ruling out inclusions: suppose that $(\mathfrak{o}(U), \mathfrak{p}(U'))$, where $\dim U$ is odd and ≥ 5, is a subpair of $(\mathfrak{sp}(V), \mathfrak{p}(V'))$. Set $n := (\dim U - 1)/2$, $n_1 := \dim U'$, $m := \dim V/2$, and

$m_1 := \dim V'$. Then $m > n$ as $\mathfrak{sp}_{2n}$ has no subalgebra isomorphic to $\mathfrak{o}(U)$. Also, by Lemma 3.4.10, $m_1 \geq n_1$. Finally, by Lemma 3.3.1, we have

$$\frac{3}{2}n_1(\frac{4}{3}n + \frac{1}{3} - n_1) = \frac{3}{2}m_1(\frac{4}{3}m + \frac{1}{3} - m_1).$$

Hence, we may apply Lemma 3.4.11 to $a_1, a_2 := \frac{3}{2}$, $b_1 := \frac{4}{3}n + \frac{1}{3}$, $b_2 := \frac{4}{3}m + \frac{1}{3}$, and $t_0 := m$, and find

$$\frac{3}{2}(\frac{2}{3}n + \frac{1}{6})^2 \geq \frac{1}{2}m(m+1),$$

from whence

$$n \geq \frac{\sqrt{3}}{2}\sqrt{m(m+1)} - \frac{1}{4} > \frac{m}{2}.$$

By Lemma 3.4.8, V splits, as a $\mathfrak{o}(U)$-module, into copies of U and a trivial module T. The inequality just derived implies that only one copy of U fits into V, so that $V \cong U \oplus T$ and $\dim T < \dim U$. But then the skew linear form on V restricts to a non-zero form on U, and this restriction is invariant under $\mathfrak{o}(U)$. However, the unique non-zero $\mathfrak{o}(U)$-invariant form on U is symmetric, not skew, and we arrive at a contradiction.   By similar arguments, the following proposition can be proved.

PROPOSITION 3.4.12. *Suppose that* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is a subpair of* $(\mathfrak{cl}(V), \mathfrak{p}(V'))$. *Then the following holds: if* $\mathfrak{cl}(V)$ *is orthogonal, then so is* $\mathfrak{cl}(U)$. *Similarly, if* $\mathfrak{cl}(V)$ *is symplectic, then so is* $\mathfrak{cl}(U)$.

We have only used half of the strength of Lemma 3.4.10 to this point; the proof of the next non-existence result exploits its full strength.

PROPOSITION 3.4.13. *Suppose that* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is a proper subpair of the pair* $(\mathfrak{sl}(V), \mathfrak{p}(V'))$. *Then* $\mathfrak{cl}(U) = \mathfrak{sp}(U)$, $\dim U = \dim V$, *and* $\dim U' = \dim V' = 1$.

PROOF. Let n be the semisimple rank of $\mathfrak{cl}(U)$, set $n_1 := \dim U'$, and write $n = n_1 + n_2$. Similarly, define m, m_1, and m_2 for $\mathfrak{sl}(V)$. Consider the four possible types of $\mathfrak{cl}(U)$:

$\mathfrak{cl}(U)$ **of type** A_n: we have $\mathrm{codim}_{\mathfrak{sl}(V)}\,\mathfrak{p}(V') = m_1(m+1-m_1)$, which is at least $n_1(n+1-n_1)$ according to Lemma 3.4.10, with equality if and only if $m_1 = n_1$ and $m + 1 - m_1 = n + 1 - n_1$, or $m_1 = n + 1 - n_1$ and $m + 1 - m_1 = n_1$, in which cases the two pairs are isomorphic.

$\mathfrak{cl}(U)$ **of type** B_n: then

$$m_1(m + 1 - m_1) \geq n_1(2n + 1 - n_1 - 1) = 2n_1n_2 + n_1^2$$

by Lemma 3.4.10. This is strictly larger than $2n_1n_2 + \binom{n_1}{2} + n_1$, the codimension of the smaller pair, unless $n_1 = 1$, in which case equality holds. But in that case, the first inequality is strict, since equality is only possible if $n_1 = n$.

$\mathfrak{cl}(U)$ **of type** C_n: then

$$m_1(m + 1 - m_1) \geq n_1(2n - n_1) = 2n_1n_2 + n_1^2$$

$$\geq 2n_1n_2 + \binom{n_1 + 1}{2} = \mathrm{codim}_{\mathfrak{cl}(U)}\,\mathfrak{p}(U'),$$

where we have equality in the first inequality if and only if $m_1 = n_2$ and $m + 1 = 2n$, and equality in the second inequality if and only if $n_1 = 1$. These are the parameters of the inclusion in the lemma.

$\mathfrak{cl}(U)$ **of type** D_n**:** then

$$m_1(m + 1 - m_1) \geq n_1(2n - n_1) = 2n_1 n_2 + n_1^2$$

by Lemma 3.4.10. This is strictly larger than $2n_1 n_2 + \binom{n_1}{2}$, the codimension of the smaller pair.

$\square$

Thus, only the inclusions among orthogonal-parabolic pairs, and those among symplectic-parabolic pairs remain to be treated.

LEMMA 3.4.14. *Suppose that* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is a proper subpair of* $(\mathfrak{cl}(V), \mathfrak{p}(V'))$*, and that* $\mathfrak{cl}(U)$ *and* $\mathfrak{cl}(V)$ *are either both orthogonal, or both symplectic.*

Then $V = U_1 \oplus T$ *as a* $\mathfrak{cl}(U)$*-module, where* U_1 *is isomorphic to* U*,* T *is trivial, and* $U_1^\perp = T$ *with respect to the bilinear form on* V*. Let* U_1' *be the image of* U' *under an isomorphism* $U \to U_1$ *of* $\mathfrak{cl}(U)$*-modules. If* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is not of type* $(B_n, \mathfrak{p}_n)$*, then* V' *splits, as a* $\mathfrak{p}(U')$*-module, into* $U_1' \oplus T'$*, where* T' *is a subspace of* T*. If* $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ *is of type* $(B_n, \mathfrak{p}_n)$*, then either* $V' = U_1' \oplus T'$ *as above, or*

$$V' = (U_1' \oplus Kv) \oplus T'$$

for some $T' \subseteq T$ *and* $v \in V$ *which has non-zero projection on both* T *and* U_1*. The* $\mathfrak{p}(U')$*-module* $U_1' \oplus Kv$ *is then isomorphic to* $(U')^\perp$*.*

PROOF. A dimension argument using Lemmas 3.4.10 and 3.4.11 shows that $\dim V < 2 \dim U$, like in the non-existence proof just preceding Proposition 3.4.12. Hence, by Lemma 3.4.8, $V = U_1 \oplus T$ for some $\mathfrak{cl}(U)$-module U_1 isomorphic to U. As $\dim(U_1) > \dim(V)/2$, the bilinear form of V restricts to a non-zero form on U_1, which must be equal, up to a constant, to the form on U_1 induced by the isomorphism $U \to U_1$. Hence, U_1 is a non-degenerate subspace of V, and $T = U_1^\perp$ as claimed.

Denote by $\pi_T : V \to T$ the projection onto T along U_1 and by $\pi_1 : V \to U_1$ the projection onto U_1 along T. Now view V as a $\mathfrak{p}(U')$-module; V' is a submodule, and so is U_1. The projection π_T induces an embedding of the quotient $V'/V' \cap U_1$ into T, hence that quotient is a trivial module. On the other hand, it projects by π_1 onto $\pi_1(V')/V' \cap U_1$, which is a submodule of $U_1/V' \cap U_1$. According to Lemma 3.3.2, there are four possibilities for $V' \cap U_1$:

(1) $V' \cap U_1 = U_1$; but then V' is invariant under $\mathfrak{cl}(U)$, which contradicts the fact that $\mathfrak{p}(U')$ is its stabilizer.

(2) $V' \cap U_1 = 0$; then V' is trivial, and so is $\pi_1(V')$. But U_1 does not have non-zero trivial submodules, so $\pi_1(V') = 0$ and again, V' is invariant under $\mathfrak{cl}(U)$.

(3) $V' \cap U_1 = U_1'$, so $\pi_1(V')/U_1'$ is trivial. But U_1/U_1' has no non-zero trivial submodules, unless $\mathfrak{cl}(U)$ is of type B_n and U' is n-dimensional. In that case, $((U_1')^\perp \cap U_1)/U_1'$ is such a submodule. This corresponds to the exceptional case in the lemma. Otherwise, $\pi_1(V') = U_1'$, which is the 'general' case.

(4) $V' \cap U_1 = (U_1')^\perp \cap U_1$, if $\mathfrak{cl}(U)$ is of type B_n and U' is n-dimensional. But $(U_1')^\perp \cap U_1$ is not totally isotropic while V' is, so this is impossible.

$\square$

We can now finish the classification of inclusions among classical-parabolic pairs.

PROPOSITION 3.4.15. *Suppose that $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is a proper subpair of the pair $(\mathfrak{cl}(V), \mathfrak{p}(V'))$, and that the classical algebras are either both orthogonal or both symplectic. Then both classical Lie algebras are orthogonal, U is $(2n+1)$-dimensional, U' is n-dimensional, V is $2n+2$-dimensional, and V' is $(n+1)$-dimensional.*

PROOF. By Lemma 3.4.14, $V = U_1 \oplus T$ as a $\mathfrak{cl}(U)$-module, where U_1 is isomorphic to U and $T = U_1^{\perp}$ is trivial. Let U_1' be the image of U' under an isomorphism $U \to U_1$ of $\mathfrak{cl}(U)$-modules. Suppose first that $(\mathfrak{cl}(U), \mathfrak{p}(U'))$ is not of type $(B_n, \mathfrak{p}_n)$. Then, again by Lemma 3.4.14, $V' = U_1' \oplus T'$ for some $T' \subseteq T$.

As the bilinear form on T is non-degenerate and $T' \subseteq V'$ is totally isotropic, we have $2 \dim T' \leq \dim T$. Let n, n_1, n_2 and m, m_1, m_2 be as in the proof of Lemma 3.4.14. Then

$$2 \dim T' = 2(m_1 - n_1)$$

and

$$\dim T = 2m + \epsilon - 2n - \delta,$$

where $\delta, \epsilon \in \{0, 1\}$ according to whether $\mathfrak{cl}(U)$ and $\mathfrak{cl}(V)$ are of type B, respectively. It follows that

$$m_2 \geq n_2 + (\delta - \epsilon)/2,$$

so that $m_2 \geq n_2$.

If both Lie algebras are of the same type, B, C or D, this together with $m_1 \geq n_1$ implies that both pairs are the same. If $\mathfrak{cl}(U)$ is of type D_n and $\mathfrak{cl}(V)$ is of type B_m, then one has $2m_1 m_2 + \binom{m_1+1}{2} > 2n_1 n_2 + \binom{n_1}{2}$: the codimensions cannot be equal. Conversely, if $\mathfrak{cl}(U)$ is of type B_n and $\mathfrak{cl}(V)$ is of type D_m, then even $m_2 \geq n_2 + 1$, and

$$2m_1 m_2 + \binom{m_1}{2} \geq 2n_1(n_2 + 1) + \binom{n_1}{2} > 2n_1 n_2 + \binom{n_1 + 1}{2}.$$

So none of these cases yield proper inclusions.

It remains to check the case where $\mathfrak{cl}(U)$ is of type B_n and U' is n-dimensional. In this case, the codimension of the larger pair equals

$$2m_1 m_2 + \binom{m_1}{2} + \epsilon m_1$$

and that of the smaller pair

$$\binom{n+1}{2},$$

and we have $m_1 \geq n$. If $\epsilon = 1$, then equality is possible only if $m_1 = n$ and $m_2 = 0$, i.e., both pairs are the same. If $\epsilon = 0$, then equality is possible only if $m_1 = n_1 + 1$, and $m_2 = 0$. This is the case mentioned in the lemma. $\square$

The Exceptional-Parabolic Pairs.

PROPOSITION 3.4.16. *Let $(\mathfrak{g}_1, \mathfrak{p}_1)$ and $(\mathfrak{g}_2, \mathfrak{p}_2)$ be primitive simple-parabolic pairs, not both of them classical-parabolic. Suppose that the former is a proper subpair of the latter. Then these pairs are those of Proposition 3.4.3.*

PROOF. For a non-classical simple Lie algebra $\mathfrak{g}$, Lemma 3.3.1 lists the codimensions of its maximal parabolic subalgebras. For each of them, calculate all primitive simple-parabolic pairs with that codimension. They turn out to be classical-parabolic pairs $(\mathfrak{cl}(V), \mathfrak{p}(V'))$. Typically, $\mathfrak{cl}(V)$ has higher rank than $\mathfrak{g}$, so that the former does not fit into the latter. Also, $\dim V$ is usually smaller than the minimal dimension of a faithful $\mathfrak{g}$-module, so that $\mathfrak{g}$ does not fit into $\mathfrak{cl}(V)$. Let us treat the exceptions to these two rules.

$\mathfrak{g}$ of type G_2: both pairs $(\mathfrak{g}, \mathfrak{p}_i)$, for $i = 1, 2$, have codimension 5, and the only other primitive simple-parabolic pair of this codimension is $(\mathfrak{o}_7, \mathfrak{p}_1)$. There is only one way to embed $\mathfrak{g}$ into $\mathfrak{o}_7$, namely via the former's irreducible module $V_{(1,0)}$. However, the parabolic subalgebra $\mathfrak{p}_2 \subseteq \mathfrak{g}$ does not stabilize a line in that module, so $(\mathfrak{g}, \mathfrak{p}_2)$ is not a subpair of $(\mathfrak{o}_7, \mathfrak{p}_1)$.

$\mathfrak{g}$ of type E_6: the pair $(\mathfrak{g}, \mathfrak{p}_2)$ has codimension 21, and the only other primitive simple-parabolic pairs of this codimension are $(\mathfrak{o}_{13}, \mathfrak{p}_6)$ and $(\mathfrak{sp}_{12}, \mathfrak{p}_6)$. Neither of these classical Lie algebras is a subalgebra of $\mathfrak{g}$; see [**19**] or the discussion of Borel-de Siebenthal subalgebras in Section 3.5. The pair $(\mathfrak{g}, \mathfrak{p}_3)$ has codimension 25, and the only other primitive simple-parabolic pair of this codimension is $(\mathfrak{o}_{27}, \mathfrak{p}_1)$. However, $\mathfrak{g}$ does not leave a symmetric form invariant in its 27-dimensional module, so the former pair is not contained in the latter. The pair $(\mathfrak{g}, \mathfrak{p}_4)$ has codimension 29, and the only other primitive simple-parabolic pair of this codimension is $(\mathfrak{sl}_{30}, \mathfrak{p}_1)$. The Lie algebra $\mathfrak{g}$ can only be embedded into $\mathfrak{sl}_{30}$ by its module $V_{\mathbf{e}_1} \oplus 3V_0$, in which $\mathfrak{p}_4$ does not leave a line invariant which is not invariant under all of $\mathfrak{g}$.

$\mathfrak{g}$ of type E_7: the pair $(\mathfrak{g}, \mathfrak{p}_1)$ has codimension 33, and the only other primitive simple-parabolic pairs of this codimension are $(\mathfrak{o}_{15}, \mathfrak{p}_6)$ and $(\mathfrak{sp}_{14}, \mathfrak{p}_6)$. But neither of these classical Lie algebras fit into $\mathfrak{g}$; see [**19**] or the discussion on Borel-de Siebenthal subalgebras in Section 3.5. The pair $(\mathfrak{g}, \mathfrak{p}_7)$ has codimension 27, and the only other primitive simple-parabolic pair of this codimension is $(\mathfrak{o}_{14}, \mathfrak{p}_3)$. However, $\mathfrak{g}$ does not have a subalgebra isomorphic to $\mathfrak{o}_{14}$; see [**19**] or the discussion of Borel-de Siebenthal subalgebras in Section 3.5. $\square$

PROOF OF THEOREM 3.4.4. Combine Lemmas 3.4.7 and 3.4.9, and Propositions 3.4.12, 3.4.13, 3.4.15, and 3.4.16. $\square$

Maximality of Simple-Parabolic Pairs. Having dealt with inclusions among simple-parabolic pairs, we now prove that such pairs cannot be subpairs of other effective pairs; this leads to the following theorem.

THEOREM 3.4.17. *Apart from the pairs $(\mathfrak{sp}_{2n}, \mathfrak{p}_1)$ for $n \geq 2$, $(\mathfrak{o}_{2n+1}, \mathfrak{p}_n)$ for $n \geq 2$, and $(G_2, \mathfrak{p}_1)$, all primitive simple-parabolic pairs are maximal among the effective pairs.*

We first prove that a primitive pair cannot be contained in an imprimitive one.

LEMMA 3.4.18. *Let $(\mathfrak{g}_1, \mathfrak{k}_1)$ be a subpair of $(\mathfrak{g}_2, \mathfrak{k}_2)$, and suppose that the former pair is primitive. Then so is the latter.*

PROOF. Denote the monomorphism by $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$. Suppose that there exists a subalgebra $\mathfrak{l}_2$ of $\mathfrak{g}_2$ with $\mathfrak{k}_2 \subsetneq \mathfrak{l}_2 \subsetneq \mathfrak{g}_2$. Then $\mathfrak{l}_1 := \phi^{-1}(\mathfrak{l}_2)$ is a subalgebra of $\mathfrak{g}_1$, and as ϕ induces an isomorphism $\mathfrak{g}_1/\mathfrak{k}_1 \to \mathfrak{g}_2/\mathfrak{k}_2$, $\mathfrak{l}_1$ lies properly in between $\mathfrak{k}_1$ and $\mathfrak{g}_1$, a contradiction. □

The following proposition shows that simple-parabolic pairs are not subpairs of other primitive pairs.

PROPOSITION 3.4.19. *Let $(\mathfrak{g}, \mathfrak{p})$ be a primitive simple-parabolic pair, let $(\mathfrak{l}, \mathfrak{k})$ be any primitive effective pair, and assume that the former is a subpair of the latter. Then $(\mathfrak{l}, \mathfrak{k})$ is also simple-parabolic.*

One can prove this using tedious dimension arguments such as the ones ruling out inclusions among classical-parabolic pairs, but we give an alternative argument using the corresponding algebraic groups.

PROOF. First suppose that $(\mathfrak{l}, \mathfrak{k})$ is of the form $(\mathfrak{k} \ltimes \mathfrak{m}, \mathfrak{k})$, where $\mathfrak{k}$ acts faithfully and irreducibly on the Abelian ideal $\mathfrak{m}$. By Proposition 3.5.1 we have a chain

$$(\mathfrak{g}, \mathfrak{p}) \subsetneq (\mathfrak{l}, \mathfrak{k}) \subsetneq (\mathfrak{sl}_{n+1}, \mathfrak{p}_1),$$

where $n := \dim \mathfrak{m}$. Hence, by Proposition 3.4.13, n is odd and $\mathfrak{g} \cong \mathfrak{sp}_{n+1}$. But then $\mathfrak{g}$ is already a maximal subalgebra of $\mathfrak{sl}_{n+1}$, a contradiction.

If $(\mathfrak{l}, \mathfrak{k})$ is not of the above form, then $\mathfrak{l}$ is semisimple by Theorem 3.2.1. Let L be the unique connected semisimple algebraic group with Lie algebra $\mathfrak{l}$ and the additional property that every $\mathfrak{l}$-module is an L-module, i.e., the *universal* connected group with Lie algebra $\mathfrak{l}$ ([**55**], page 45). Similarly, define G for $\mathfrak{g}$. Then ϕ lifts to an embedding $\pi : G \to L$ of algebraic groups such that $d_e\pi = \phi$ ([**50**], Theorems 1.2.6 and 3.3.4). Denote by P and H the (Zariski) closed connected subgroups of G and L with Lie algebras $\mathfrak{p}$ and $\mathfrak{k}$, respectively. Now $\pi(P)$ is a closed connected subgroup of L ([**5**], Corollary 1.4) with Lie algebra $\phi(\mathfrak{p}) \subseteq \mathfrak{k}$, so $\pi(P) \subseteq H$. By the universal property of the quotient G/P ([**5**], §6), π induces a morphism

$$\bar{\pi} : G/P \to L/H,$$

whose differential at eP equals the induced map

$$\bar{\phi} : \mathfrak{g}/\mathfrak{p} \to \mathfrak{l}/\mathfrak{k}.$$

As $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, P is a parabolic subgroup of G, and G/P is a complete variety ([**5**], Corollary 11.2). Hence, its image under $\bar{\pi}$ is closed in L/H. Moreover, as $d_{eP}(\bar{\pi}) = \bar{\phi}$ is a linear isomorphism, $\bar{\pi}(G/P)$ has the same dimension as the irreducible variety L/H. It follows that $\bar{\pi}$ is surjective, and that L/H is complete. But then H is a parabolic subgroup of L, again by [**5**], Corollary 11.2. Hence, $\mathfrak{k}$ is a parabolic subalgebra of $\mathfrak{l}$, as claimed. Finally, we deduce from Theorem 3.2.1 that $\mathfrak{l}$ must be simple for $(\mathfrak{l}, \mathfrak{k})$ to be a primitive pair. □

PROOF OF THEOREM 3.4.17. Combine Theorem 3.4.4, Lemma 3.4.18, and Proposition 3.4.19. □

3.5. Other Embeddings into Simple-Parabolic Pairs

Now that we know all inclusions among simple-parabolic pairs, we proceed to investigate which other primitive pairs can be embedded into simple-parabolic pairs. This section is organized as follows: we run through Morozov's and Dynkin's classifications, and determine for each entry $(\mathfrak{g}, \mathfrak{k})$ whether or not it is a subpair of a simple-parabolic pair.

$\mathfrak{g}$ **is not Simple.** Morozov's classification starts with the following type of primitive pairs.

PROPOSITION 3.5.1. *The pair* $(\mathfrak{k} \ltimes \mathfrak{m}, \mathfrak{k})$, *where* $\mathfrak{k}$ *acts faithfully and irreducibly on the Abelian ideal* $\mathfrak{m}$, *is a subpair of*

$$(\mathfrak{sl}(\mathfrak{m} \oplus Kv), \mathfrak{p}(Kv)),$$

where Kv *denotes an auxiliary one-dimensional vector space spanned by* v.

PROOF. Consider the linear map $\phi : \mathfrak{k} \ltimes \mathfrak{m} \to \mathfrak{sl}(\mathfrak{m} \oplus Kv)$ determined by

$$\phi(X)Y = [X, Y], \qquad\qquad \phi(X)v = 0,$$
$$\phi(Y_1)Y_2 = 0, \text{ and} \qquad\qquad \phi(Y)v = Y$$

for all $X \in \mathfrak{k}$ and $Y, Y_1, Y_2 \in \mathfrak{m}$. Then ϕ is a monomorphism of Lie algebras, and it is clear that

$$\phi^{-1}(\mathfrak{p}(Kv)) = \mathfrak{k}.$$

Furthermore, both pairs have codimension $\dim \mathfrak{m}$. $\square$

The second entry in Morozov's list comprises the 'diagonal pairs'. The following proposition shows that some of these are subpairs of classical-parabolic pairs.

PROPOSITION 3.5.2. *Let* $\mathfrak{d} = \{(X, X) \mid X \in \mathfrak{sl}(V)\}$ *be the diagonal subalgebra of* $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$. *Then the pair* $(\mathfrak{sl}(V) \oplus \mathfrak{sl}(V), \mathfrak{d})$ *is a subpair of the pair* $(\mathfrak{sl}(V \otimes V^*), \mathfrak{p})$, *where* $\mathfrak{p}$ *is the parabolic subalgebra leaving a suitable one-dimensional subspace of* $V \otimes V^*$ *invariant.*

PROOF. We obtain an embedding ϕ of $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$ into $\mathfrak{sl}(V \otimes V^*)$ by viewing $V \otimes V^*$ as a module for the former algebra. We may identify $V \otimes V^*$ with $\mathfrak{gl}(V)$. Under this identification, we have for $X \in \mathfrak{sl}(V)$, $v_0, v \in V$, and $f_0 \in V^*$:

$$(X, X)(v_0 \otimes f_0)(v) = (Xv_0 \otimes f_0 + v_0 \otimes Xf_0)(v)$$
$$= f_0(v)Xv_0 - f_0(Xv)v_0$$
$$= [X, v_0 \otimes f_0]v.$$

Hence, considered as an $\mathfrak{sl}(V)$-module through the isomorphism $X \mapsto (X, X)$ from $\mathfrak{sl}(V)$ onto $\mathfrak{d}$, the space $V \otimes V^*$ is isomorphic to $\mathfrak{gl}(V)$ with the adjoint action. Let I be the element of $V \otimes V^*$ corresponding to the identity of $\mathfrak{gl}(V)$; then I spans a trivial $\mathfrak{d}$-submodule. Set

$$\mathfrak{p} := \{X \in \mathfrak{sl}(V \otimes V^*) \mid XI \in KI\}.$$

Then, by maximality of $\mathfrak{d}$ in $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V^*)$, we have

$$\mathfrak{d} = \phi^{-1}(\mathfrak{p}).$$

Finally, the codimensions of both pairs equal $\dim(V)^2 - 1$. $\square$

By this proposition, the pair $(\mathfrak{sl}(V) \oplus \mathfrak{sl}(V), \mathfrak{d})$ has a polynomial transitive realization; compare this to Example 2.5.4. A more geometrical description of the above situation is the following: $\mathrm{SL}(V) \times \mathrm{SL}(V)$ acts on $\mathbb{P}(\mathfrak{gl}(V))$ by

$$(a,b)M := aMb^{-1}, \ a,b \in \mathrm{SL}(V) \text{ and } M \in \mathfrak{gl}(V)$$

in homogeneous coordinates. There is one open dense orbit in the Zariski topology, namely that of I. The stabilizer D of I is the diagonal subgroup of $\mathrm{SL}(V) \times \mathrm{SL}(V)$, and we have thus embedded the homogeneous space $\mathrm{SL}(V) \times \mathrm{SL}(V)/D$ into the projective space $\mathbb{P}(\mathfrak{gl}(V))$.

The following proposition shows that other diagonal pairs cannot be embedded into classical-parabolic pairs.

PROPOSITION 3.5.3. *Suppose that the diagonal pair $(\mathfrak{k}_1 \oplus \mathfrak{k}_2, \mathfrak{k})$ is a subpair of the classical-parabolic pair $(\mathfrak{cl}(V), \mathfrak{p}(V'))$. Then $\mathfrak{k} \cong \mathfrak{sl}_n$, $\mathfrak{cl}(V) \cong \mathfrak{sl}_{n^2}$, and V' is one-dimensional.*

PROOF. View V as a $\mathfrak{k}_1 \oplus \mathfrak{k}_2$-module through the embedding $\mathfrak{k}_1 \oplus \mathfrak{k}_2 \to \mathfrak{cl}(V)$. As such it is a direct sum of modules of the form $V_1 \otimes V_2$, where V_i is an irreducible $\mathfrak{k}_i$-module. If for all these submodules either V_1 or V_2 is trivial, then any $\mathfrak{k}$-invariant subspace is invariant under $\mathfrak{k}_1 \oplus \mathfrak{k}_2$. In particular, this is the case for V', a contradiction. Hence, V must contain a $\mathfrak{k}_1 \oplus \mathfrak{k}_2$-submodule of the form $V_1 \otimes V_2$, where V_1 and V_2 are both non-trivial, hence faithful.

From this point, a complete proof proceeds with a case-by-case analysis according to the Cartan type of $\mathfrak{g}$. Rather than treating all cases in detail, I first describe the arguments that apply to each of them, and then work out one particular instance.

By Lemma 3.3.3, V_1 and V_2 have dimensions of at least roughly $c\sqrt{k}$, where k is the dimension of $\mathfrak{k}$, and $c = 1$ if $\mathfrak{k}$ is of type A and $c > 1$ otherwise. On the other hand, the product of their dimensions is at most $\dim V$, which, in turn, is at most $k + 2$ by Lemma 3.4.5. This shows that the lower bound on the dimensions of V_1 and V_2 and the upper bound on the dimension of V are both rather tight, if not contradictory. An analysis of the relevant exceptional cases establishes the result.

For example, suppose that $\mathfrak{k}_1 = \mathfrak{k}_2 = \mathfrak{o}_n$ for $n \geq 6$. Then V_1 and V_2 have dimensions at least n, so that V has dimension at least n^2, and $\mathrm{codim}_{\mathfrak{cl}(V)} \mathfrak{p}(V') \geq n^2 - 2$ by Lemma 3.4.5. On the other hand,

$$\mathrm{codim}_{\mathfrak{k}_1 \oplus \mathfrak{k}_1} \mathfrak{k} = \dim \mathfrak{k} = \binom{n}{2} = n^2/2 - n/2 < n^2 - 2.$$

The other cases can be treated similarly. $\square$

$\mathfrak{g}$ is Classical Simple. Consider a primitive pair $(\mathfrak{cl}(U), \mathfrak{k})$. First, suppose that $\mathfrak{k}$ acts reducibly on U. The case where $\mathfrak{k}$ is parabolic is treated in Section 3.4. The following two propositions treat the remaining two cases.

PROPOSITION 3.5.4. *Let V be a vector space equipped with a non-degenerate symmetric bilinear form $(.,.)$, and let U be a non-degenerate subspace of V. Then the pair $(\mathfrak{o}(V), \mathfrak{o}(U) \oplus \mathfrak{o}(U^{\perp}))$ is a subpair of $(\mathfrak{sl}(V), \mathfrak{p}(U))$.*

PROOF. The embedding of $\mathfrak{o}(V)$ into $\mathfrak{sl}(V)$ is the natural one, and $\mathfrak{o}(U) \oplus \mathfrak{o}(U^{\perp})$ is precisely the pre-image of $\mathfrak{p}(U)$ under this embedding. Finally, the codimensions of both pairs equal $\dim(U) \cdot \dim(U^{\perp})$. $\square$

This inclusion has a geometric interpretation: for V and U as in the proposition, define $d := \dim U$. Then $\mathrm{O}(V)$ acts naturally on the Grassmannian variety $G_d(V)$ of all d-dimensional subspaces of V. Under this action, the orbit of U is the open dense subset of $G_d(V)$ consisting of all non-degenerate d-dimensional subspaces of V, and the stabilizer of U is $\mathrm{O}(U) \times \mathrm{O}(U^\perp)$. We thus find an open dense embedding $\mathrm{O}(V)/(\mathrm{O}(U) \times \mathrm{O}(U^\perp)) \to G_d(V)$, and the right-hand side can be identified with $\mathrm{SL}(V)/\mathrm{P}(U)$. A similar proposition holds in the symplectic case, and both the proof and the interpretation are similar.

PROPOSITION 3.5.5. *Let V be a vector space equipped with a non-degenerate skew bilinear form $(.,.)$, and let U be a non-degenerate subspace of V. Then the primitive pair $(\mathfrak{sp}(V), \mathfrak{sp}(U) \oplus \mathfrak{sp}(U^\perp))$ is a subpair of $(\mathfrak{sl}(V), \mathfrak{p}(U))$.*

Next, suppose that $\mathfrak{k}$ acts irreducibly on the standard module for $\mathfrak{g}$. We have two families of inclusions.

PROPOSITION 3.5.6. *Let V be a vector space equipped with a non-degenerate symmetric bilinear form $b = (.,.) \in S^2(V)^*$. Then $(\mathfrak{sl}(V), \mathfrak{o}(V))$ is a subpair of the classical-parabolic pair $(\mathfrak{sl}(S^2(V)^*), \mathfrak{p}(Kb))$.*

PROOF. We have
$$\mathfrak{o}(V) = \{X \in \mathfrak{sl}(V) \mid Xb = 0\}.$$
In fact, as $\mathfrak{o}(V)$ is maximal in $\mathfrak{sl}(V)$, it is equal to the seemingly larger algebra
$$\{X \in \mathfrak{sl}(V) \mid Xb \in Kb\}.$$
Let $\phi : \mathfrak{sl}(V) \to \mathfrak{sl}(S^2(V)^*)$ be the embedding given by the $\mathfrak{sl}(V)$-module structure on $S^2(V)^*$; then we have
$$\mathfrak{o}(V) = \phi^{-1}(\mathfrak{p}(Kb)).$$
The codimension of $\mathfrak{o}(V)$ in $\mathfrak{sl}(V)$ equals $(m^2 - 1) - m(m-1)/2 = m(m+1)/2 - 1$, and that of $\mathfrak{p}$ in $\mathfrak{sl}(S^2(V)^*)$ equals $m(m-1)/2 + m - 1$, which is the same. $\square$

The geometric interpretation is the following: on the projective space $\mathbb{P}(S^2(V)^*)$ of all symmetric bilinear forms, the group $\mathrm{SL}(V)$ has an open dense orbit, consisting of the non-degenerate forms. This yields an open dense embedding $\mathrm{SL}(V)/\mathrm{O}(V) \to \mathrm{SL}(S^2(V^*))/\mathrm{P}(Kb)$. A similar statement, with a similar proof and interpretation, holds in the symplectic case.

PROPOSITION 3.5.7. *Let V be a vector space equipped with a non-degenerate skew bilinear form $b = (.,.) \in \Lambda^2(V)^*$. Then $(\mathfrak{sl}(V), \mathfrak{sp}(V))$ is a subpair of the classical-parabolic pair $(\mathfrak{sl}(\Lambda^2(V)^*), \mathfrak{p}(Kb))$.*

PROPOSITION 3.5.8. *The pair $(\mathfrak{o}_7, G_2)$ is a subpair of $(\mathfrak{sl}_8, \mathfrak{p}_1)$.*

PROOF. Embed $\mathfrak{o}_7$ into $\mathfrak{sl}_8$ by means of its irreducible spin module with highest weight $\mathbf{e}_3$. Restricted to G_2, this module has a trivial one-dimensional submodule Kv. By maximality of G_2 in $\mathfrak{o}_7$, the former is the stabilizer of Kv in the latter. A check that both pairs have codimension 7 concludes the proof. $\square$

For a geometric interpretation of this inclusion see [**1**]; that paper classifies the pairs (G, X), where G is a complex linear algebraic group acting morphically on the smooth complete algebraic variety X with an open orbit Ω such that $X \setminus \Omega$ is a single orbit of codimension 1. Akhiezer calls such a variety X a *two-orbit variety*.

THEOREM 3.5.9. *Let $\mathfrak{k}$ be a semisimple Lie algebra, and let U be a faithful irreducible $\mathfrak{k}$-module. Let $\mathfrak{cl}(U)$ be $\mathfrak{o}(U)$, $\mathfrak{sp}(U)$, or $\mathfrak{sl}(U)$, according to whether $\mathfrak{k}$ leaves invariant a symmetric, a skew, or no non-degenerate bilinear form on U, and assume that $(\mathfrak{cl}(U), \mathfrak{k})$ is primitive. Suppose that $(\mathfrak{cl}(U), \mathfrak{k})$ is a subpair of a classical-parabolic pair $(\mathfrak{cl}(V), \mathfrak{p}(V'))$. Then either*

(1) *$\mathfrak{k}$ is of type G_2, $U = V_{\mathbf{e}_1}$ is 7-dimensional, $\mathfrak{cl}(U) = \mathfrak{o}(U)$, V is the 8-dimensional spin module of $\mathfrak{cl}(U)$, $\mathfrak{cl}(V) = \mathfrak{sl}(V)$, and V' is a one-dimensional subspace of V, or*

(2) *$\mathfrak{k} \cong \mathfrak{sp}(U_2) \oplus \mathfrak{sp}(U_4)$, where $\dim U_i = i$, $U = U_2 \otimes U_4$, $\mathfrak{cl}(U) \cong \mathfrak{o}(U)$, V is one of the 8-dimensional spin modules of $\mathfrak{o}(U)$, $\mathfrak{cl}(V) = \mathfrak{sl}(V)$, and V' is a 3-dimensional subspace of V.*

The proof relies on the following lemma.

LEMMA 3.5.10. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, V a finite-dimensional $\mathfrak{g}$-module, and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$. Suppose that all irreducible $\mathfrak{g}$-submodules of V restrict to irreducible $\mathfrak{k}$-modules, and that two such restrictions are isomorphic as $\mathfrak{k}$-modules if and only if they are isomorphic as $\mathfrak{g}$-modules. Then any $\mathfrak{k}$-submodule of V is invariant under $\mathfrak{g}$.*

PROOF. As $\mathfrak{g}$ is semisimple, V has a unique decomposition

$$V = \bigoplus_{i=1}^{k} V_i$$

into $\mathfrak{g}$-isotypical components. By assumption, this is also a decomposition into $\mathfrak{k}$-isotypical components. Now it can be shown that each $\mathfrak{k}$-submodule V' of V decomposes as

$$V' = \bigoplus_{i=1}^{k} (V' \cap V_i),$$

and that each component $V' \cap V_i$ is in fact a $\mathfrak{g}$-module. $\square$

PROOF OF THEOREM 3.5.9. View V as a $\mathfrak{cl}(U)$-module through the embedding $\mathfrak{cl}(U) \to \mathfrak{cl}(V)$. Write $V = \bigoplus_i V_i$, where the V_i are irreducible $\mathfrak{cl}(U)$-modules, and denote by π_i the projection from V onto V_i along $\bigoplus_{j \neq i} V_j$. Now $V' \subseteq V$ is invariant under $\mathfrak{k}$, but not under all of $\mathfrak{cl}(U)$. Hence by Lemma 3.5.10 there exists an i such that the $\mathfrak{k}$-submodule $\pi_i(V')$ of V_i is non-zero, and V_i is neither trivial nor isomorphic to U.

Let n be the semisimple rank of $\mathfrak{cl}(U)$. From this point we proceed according to the Cartan type of $\mathfrak{cl}(U)$, leaving out the tedious case-by-case analysis needed in small dimensions.

First suppose that $\mathfrak{cl}(U)$ is of type B_n for $n \geq 7$. Then $\dim V \geq \dim V_i \geq \binom{2n+1}{2}$ by Lemma 3.3.3. But then $\mathrm{codim}_{\mathfrak{o}(V)} \mathfrak{p}(V') \geq \binom{2n+1}{2} - 2$ by Lemma 3.4.5, although $\mathrm{codim}_{\mathfrak{o}(U)} \mathfrak{k}$ is at most $\dim \mathfrak{o}(U) - \dim \mathfrak{sl}_2 = \binom{2n+1}{2} - 3$, a contradiction. The cases $n = 2, \ldots, 6$ can be treated one by one, and give rise to the first inclusion in the theorem.

If $\mathfrak{cl}(U)$ is of type D_n, then a similar argument leads to non-existence of inclusions other than the second one in the theorem.

Next suppose that $\mathfrak{cl}(U)$ is of type C_n. Then V_i is either isomorphic to $S^2(U)$, or isomorphic to $\Lambda^2(U)$ minus a one-dimensional trivial module, or even larger. In the last case, we find a lower bound on $\dim V$, whence on $\operatorname{codim}_{\mathfrak{cl}(V)}\mathfrak{p}(V')$ by Lemma 3.4.5, of degree at least 3 in n, while $\operatorname{codim}_{\mathfrak{sp}(U)}\mathfrak{k} \leq \binom{2n+1}{2} - 3$. This is a contradiction unless n is very small, and a careful analysis shows that there are no inclusions for small n either. Suppose now that V_i is isomorphic to $S^2(U)$. As the binary form on U invariant under $\mathfrak{k}$ is unique up to a scalar—and skew by assumption—$\mathfrak{k}$ does not leave a line invariant in V_i. It follows that $\dim V' \geq 2$. This gives a lower bound on $\operatorname{codim}_{\mathfrak{cl}(V)}\mathfrak{p}(V')$ of roughly $2\dim V$, which is at least $2\binom{2n+1}{2}$. Again, this is larger than the $\operatorname{codim}_{\mathfrak{sp}(U)}\mathfrak{k}$. The same argument applies when V_i is isomorphic to $\Lambda^2(U) - K$.

Finally suppose that $\mathfrak{cl}(U)$ is of type A_n. As $\mathfrak{k}$ is supposed not to leave invariant any bilinear form on U, its dimension is at least $\dim \mathfrak{sl}_3 = 8$. Moreover, any non-trivial irreducible $\mathfrak{k}$-module has dimension at least 3, and U and U^* are non-isomorphic as $\mathfrak{k}$-modules. Hence by Lemma 3.5.10, we may assume that V_i is not isomorphic to U^* either. It follows that V_i is equivalent to $S^2(U)$, or equivalent to $\Lambda^2(U)$, or even larger. In the first two cases, $\mathfrak{k}$ does not leave a line invariant in V_i, so $\dim V' \geq 3$. By Lemma 3.3.1, this leads to a lower bound on $\operatorname{codim}_{\mathfrak{cl}(V)}\mathfrak{p}(V')$ of roughly $3\dim V$, which is at least $3\binom{n+1}{2}$, while $\operatorname{codim}_{\mathfrak{sl}(U)}\mathfrak{k} \leq (n+1)^2 - 1 - 8$. This is a contradiction unless n is very small, and those few cases are easily handled one by one. If V_i is not equivalent to $S^2(U)$ or $\Lambda^2(U)$, then we find a lower bound on $\dim V$, whence on $\operatorname{codim}_{\mathfrak{cl}(V)}\mathfrak{p}(V')$ by Lemma 3.4.5, of degree three in n. Again, this contradicts the fact that $\operatorname{codim}_{\mathfrak{sl}(U)}\mathfrak{k}$ has an upper bound quadratic in n. $\square$

REMARK 3.5.11. For $i = 2, 4$, let U_i be an i-dimensional vector space equipped with a non-degenerate skew bilinear form. Then the Lie algebra $\mathfrak{o}(U_2 \otimes U_4)$ has an outer automorphism mapping the subalgebra $\mathfrak{sp}(U_2) \oplus \mathfrak{sp}(U_4)$ to $\mathfrak{o}(U_3) \oplus \mathfrak{o}(U_3^\perp)$ for some 3-dimensional non-degenerate subspace U_3 of $U_2 \otimes U_4$. Hence, the pair $(\mathfrak{o}(U_2 \otimes U_4), \mathfrak{sp}(U_2) \oplus \mathfrak{sp}(U_4))$ is isomorphic to the pair $(\mathfrak{o}(U_2 \otimes U_4), \mathfrak{o}(U_3) \oplus \mathfrak{o}(U_3^\perp))$, which is a subpair of $(\mathfrak{sl}(U_2 \otimes U_4), \mathfrak{p}(U_3))$ by Proposition 3.5.4.

The following conjecture would conclude this subsection's quest.

CONJECTURE 3.5.12. *Let $(\mathfrak{cl}(U), \mathfrak{k})$ be as in Theorem 3.5.9. Then it is not a subpair of any exceptional simple-parabolic pair.*

Let me outline how I would go about proving this with help of a computer: for each exceptional simple $\mathfrak{g}$ and for each node i in the Dynkin diagram of $\mathfrak{g}$, the codimension of $\mathfrak{p}_i$ in $\mathfrak{g}$ is listed in Lemma 3.3.1. Compute, using my program in LiE, all primitive pairs $(\mathfrak{cl}(U), \mathfrak{k})$ as in Theorem 3.5.9 of this codimension; usually there are only few such pairs. It may happen that the orbit of $\mathfrak{k}$ under $\operatorname{Aut}(\mathfrak{cl}(U))$ is larger than the orbit of $\mathfrak{k}$ under $\operatorname{Inn}(\mathfrak{cl}(U))$, the group of inner automorphisms of $\mathfrak{cl}(U)$. Fix representatives $\mathfrak{k}_1, \ldots, \mathfrak{k}_l$ for each of the $\operatorname{Inn}(\mathfrak{cl}(U))$-orbits contained in $\operatorname{Aut}(\mathfrak{cl}(U))\mathfrak{k}$. In LiE this is done by choosing *restriction matrices*, which describe the linear maps $\mathfrak{h}^* \to \mathfrak{h}_j^*$ dual to the inclusions $\mathfrak{h}_j \to \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{cl}(U)$ containing the Cartan subalgebra $\mathfrak{h}_j$ of $\mathfrak{k}_j$. This step is needed, because $\mathfrak{cl}(U)$-modules may split differently when restricted to different $\mathfrak{k}_j$.

Now compute all embeddings of $\mathfrak{cl}(U)$ into $\mathfrak{g}$, up to inner automorphisms of the latter algebra. Each such embedding defines the structure of a $\mathfrak{cl}(U)$-module on the

$\mathfrak{g}$-module $V_{\mathbf{e}_i}$; let t be the number of one-dimensional trivial $\mathfrak{cl}(U)$-submodules in it. Similarly, the inclusion $\mathfrak{k}_j \to \mathfrak{cl}(U)$ endows $V_{\mathbf{e}_i}$ with the structure of a $\mathfrak{k}_j$-module; let s_j be the number of one-dimensional trivial $\mathfrak{k}_j$-modules in it. Now clearly $s_j \geq t$, and if the embedding $\mathfrak{cl}(U) \to \mathfrak{g}$ is to define an inclusion of $(\mathfrak{cl}(U), \mathfrak{k}_j)$ into $(\mathfrak{g}, \mathfrak{p}_i)$, then s_j must be strictly larger than t. Although this seems to happen only rarely, the following two examples show that it does.

(1) Let $\mathfrak{g}$ be of type E_7, $i = 6$, and $(\mathfrak{cl}(U), \mathfrak{k}) = (\mathfrak{o}_{12}, \mathfrak{sp}_2 \otimes \mathfrak{sp}_6)$, of codimension 42. Here $l = 1$, there is only one embedding of $\mathfrak{o}_{12}$ into $\mathfrak{g}$, $t = 1$, and $s_1 = 2$.

(2) Let $\mathfrak{g}$ be of type E_8, $i = 2$, and $(\mathfrak{cl}(U), \mathfrak{k}) = (\mathfrak{o}_{15}, \mathfrak{o}_3 \otimes \mathfrak{o}_5)$, of codimension 92. Here, too, $l = 1$, there is only one embedding of $\mathfrak{o}_{15}$ into $\mathfrak{g}$, $t = 1$, and $s_1 = 4$.

Thus, the proof of Conjecture 3.5.12 reduces to ruling out candidate inclusions such as these two.

$\mathfrak{g}$ is Exceptional Simple. For the case where $\mathfrak{k}$ is reductive, we have the following inclusions, both of which correspond to two-orbit varieties [**1**].

PROPOSITION 3.5.13. *The pair $(G_2, \mathfrak{sl}_3)$ is a subpair of $(\mathfrak{sl}_7, \mathfrak{p}_1)$.*

PROOF. Embed G_2 into $\mathfrak{sl}_7$ by its 7-dimensional irreducible module. The restriction of this representation to $\mathfrak{sl}_3$ contains a one-dimensional trivial module Kv. Hence by its maximality in $\mathfrak{g}$, the subalgebra $\mathfrak{sl}_3$ is the stabilizer, in $\mathfrak{g}$, of Kv. A check that both codimensions equal 6 finishes the proof. □

PROPOSITION 3.5.14. *The pair $(F_4, \mathfrak{o}_9)$ is a subpair of $(E_6, \mathfrak{p}_1)$, where $\mathfrak{p}_1$ is the maximal parabolic subalgebra of E_6 corresponding to the first node.*

I outline how to check this proposition with de Graaf's algorithms in `GAP`: first construct the simple Lie algebra of type E_6. Table 24 of [**19**] contains explicit expressions for the simple root vectors of the non-regular subalgebra F_4 of E_6 as linear combinations of the root vectors of E_6. With these, construct the required embedding of F_4 into E_6 in `GAP`. Next, construct B_4 inside F_4 by taking three of the latter's simple root vectors, plus the highest root vector, as well as their opposites. Now calculate the common zero space of B_4 inside the E_6-module $V_{\mathbf{e}_1}$. This two-dimensional space happens to be spanned by a one-dimensional trivial F_4-module, and an E_6-weight vector v which is not a zero vector of F_4. Now, as $V_{\mathbf{e}_1}$ is a minuscule module ([**8**], Chapitre 8), Kv is the highest weight line with respect to some choice of simple roots, so that its stabilizer in E_6 is conjugate to $\mathfrak{p}_1$. We thus find that B_4 is the pre-image in F_4 of a conjugate of $\mathfrak{p}_1$ under the embedding $F_4 \to E_6$ constructed above. As both codimensions equal 16, the proposition follows.

In Propositions 3.5.13 and 3.5.14, the smaller of the two pairs is of the form $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is simple, and $\mathfrak{k}$ is semisimple of the same rank as $\mathfrak{g}$. In this case $\mathfrak{k}$ is called a *Borel-de Siebenthal subalgebra* of $\mathfrak{g}$, and we shall call $(\mathfrak{g}, \mathfrak{k})$ a *Borel-de Siebenthal pair*. It is well known how to obtain all Dynkin diagrams of *maximal* Borel-de Siebenthal subalgebras of a semisimple Lie algebra $\mathfrak{g}$: simply leave out a vertex from the extended Dynkin diagram of $\mathfrak{g}$. For $\mathfrak{sl}_n$ this yields only $\mathfrak{sl}_n$ itself, for $\mathfrak{o}_n$ all subalgebras of the form $\mathfrak{o}_{n_1} \oplus \mathfrak{o}_{n_2}$ with $n_1 + n_2 = n$, and for $\mathfrak{sp}_{2n}$ the subalgebras of the form $\mathfrak{sp}_{2n_1} \oplus \mathfrak{sp}_{2n_2}$ with $n_1 + n_2 = n$. In the latter two cases, primitive Borel-de Siebenthal pairs are subpairs of simple-parabolic pairs by Propositions 3.5.4 and 3.5.5. The two inclusions above

may lead one to conjecture that all primitive Borel-de Siebenthal pairs are subpairs of simple-parabolic pairs, but this is far from true.

THEOREM 3.5.15. *The only inclusions of a primitive exceptional Borel-de Siebenthal pair into a simple-parabolic pair are the ones of Propositions 3.5.13 and 3.5.14.*

This theorem can be proved using my `LiE` program and arguments similar to those motivating Conjecture 3.5.12.

Integration to Algebraic Group Actions

4.1. Introduction

In Chapter 2 we computed various realizations of finite-dimensional pairs $(\mathfrak{g}, \mathfrak{k})$ with formal power series coefficients. In some cases, we could actually do with polynomial coefficients, and the inclusions of Chapter 3 are a source of even more polynomial realizations. Many of these polynomially realizable pairs are *algebraic*, i.e., $\mathfrak{g}$ is the Lie algebra of an affine algebraic group G, and $\mathfrak{k}$ is the Lie algebra of a (Zariski) closed subgroup H of G. The chapter at hand gives an *a priori* explanation why such pairs should have nice realizations.

Let G be an affine algebraic group over an algebraically closed field K of characteristic 0. Denote the unit of G by e, and the stalk at e of the sheaf of regular functions on open subsets of G by $\mathcal{O}_e$. To keep notation consistent with the other chapters, we denote general Lie algebras by lowercase German letters. However, to stress the dependence on the algebraic group G, we write $L(G)$ for the Lie algebra of G, which coincides with $T_e(G) = \mathrm{Der}_K(\mathcal{O}_e, K)$ as a vector space. Let V be an algebraic variety, and $\alpha : G \times V \to V$ a morphic action of G on V, i.e., an action that is also a morphism of algebraic varieties. Then we can 'differentiate' α to a representation of $L(G)$ by derivations on $K[U]$, for any open affine subset U of V. Assuming that U is clear from the context, this representation is denoted by $X \mapsto -X *_\alpha$; its construction, and the presence of the minus sign, is explained on page 58.

As a special case, take $V := G/H$, where H is a closed subgroup of G. The group G acts on V by $\alpha(g_1, g_2 H) := g_1 g_2 H$. Let U be an affine open neighbourhood of $p := eH$ in V. Then $(X *_\alpha f)(p) = 0$ for all $f \in K[U]$ if and only if $X \in L(H)$. Passing to the completion of the local ring $\mathcal{O}_p$ at p, we find a transitive realization of the pair $(L(G), L(H))$ into $\hat{\mathfrak{D}}^{(\dim V)}$, whose coefficients are algebraic functions. For example, if G is connected and semisimple, and H is parabolic, then eH has an open neighbourhood in G/H that is isomorphic to an affine space, whence it follows that $(L(G), L(H))$ has a polynomial realization. Similar arguments are applied to other algebraic pairs in Section 4.3.

The bulk of this chapter, however, deals with a converse of the above construction: given a finite-dimensional Lie algebra $\mathfrak{l}$ and a homomorphism $\rho : \mathfrak{l} \to \mathrm{Der}_K(K[U])$ for some affine algebraic variety U, can we find an affine algebraic group G, an algebraic variety V containing U as an open dense subset, an action $\alpha : G \times V \to V$, and an embedding $\mathfrak{l} \to L(G)$ such that ρ is the restriction to $\mathfrak{l}$ of the homomorphism $X \mapsto -X *_\alpha$?

The two main results of this chapter answer this question affirmatively for many interesting cases. First, if G is to act on U itself, the action of $\mathfrak{l}$ on $K[U]$ must be locally

finite (see page 57 for the precise definition). Conversely, we have the following theorem for ρ an inclusion.

THEOREM 4.1.1. *Let U be an affine algebraic variety, and let $\mathfrak{l}$ be a locally finite Lie subalgebra of $\mathrm{Der}_K(K[U])$. Then there exist a linear algebraic group G, a morphic action $G \times U \to U$, and an embedding $\mathfrak{l} \to L(G)$ such that the representation $X \mapsto -X*_\alpha$ restricts to the identity on $\mathfrak{l}$.*

Note that we do not require $\mathfrak{l}$ to be the Lie algebra of an algebraic group. Indeed, in Example 4.4.3 we shall see that $\mathfrak{l}$ need not coincide with $L(G)$.

As an example, let U be the affine line, with coordinate function x. Then the derivation ∂_x acts locally nilpotently on $K[U] = K[x]$, whence locally finitely. The derivation $x\partial_x$ acts semisimply on $K[U]$, whence also locally finitely. On the other hand, the derivation $x^2\partial_x$ does not act locally finitely. Theorem 4.1.1 can therefore be applied to $\langle \partial_x, x\partial_x \rangle_K$, but not to $\langle \partial_x, x\partial_x, x^2\partial_x \rangle_K$. However, any differential equation of the form

$$x'(t) = \lambda + \mu x(t) + \nu x(t)^2, \quad x(0) = x_0$$

with $\lambda, \mu, \nu \in K$ has a solution which is a rational expression in x_0, t, and $\exp(\alpha t)$ for some $\alpha \in K$. This observation is a key to our results in the case that ρ is not locally finite.

More formally, we introduce the exponential map. For simplicity, let us assume that U is irreducible, so that $K[U]$ is an integral domain with field $K(U)$ of fractions. Let t be a variable, and denote by $K[U][[t]]$ the algebra of formal power series in t with coefficients from $K[U]$. For $f_1, \ldots, f_k \in K[U][[t]]$, we denote by $K(U)(f_1, \ldots, f_k)$ the subfield of the field of fractions of $K[U][[t]]$ generated by the f_i. For $\nabla \in \mathrm{Der}_K(K[U])$, we define the map $\exp(t\nabla)$ from $K[U]$ to $K[U][[t]]$ as follows:

$$(8) \qquad \exp(t\nabla)f = \sum_{n=0}^{\infty} \frac{t^n}{n!} \nabla^n(f), \quad f \in K[U].$$

Here, we only mention two consequences of our second main result (which is Theorem 4.5.4).

THEOREM 4.1.2. *Let $\mathfrak{l}$ be a nilpotent Lie algebra, U an irreducible affine algebraic variety, $\rho : \mathfrak{l} \to \mathrm{Der}_K(K[U])$ a Lie algebra homomorphism, and $X_1, \ldots, X_k$ a basis of $\mathfrak{l}$ such that $\langle X_i, \ldots, X_k \rangle$ is an ideal in $\mathfrak{l}$ for all $i = 1, \ldots, k$.*
Suppose that ρ satisfies

$$\exp(t\rho(X_i))K[U] \subseteq K(U)(t)$$

for all i. Then there exist a connected linear algebraic group G having $\mathfrak{l}$ as its Lie algebra, an algebraic variety V containing U as an open dense subset, and a morphic action $\alpha : G \times V \to V$ such that the corresponding representation $X \mapsto -X*_\alpha$, $\mathfrak{l} \to \mathrm{Der}_K(K[U])$ coincides with ρ.

THEOREM 4.1.3. *Let $\mathfrak{l}$ be a semisimple Lie algebra of Lie rank l, U an irreducible affine algebraic variety, and $\rho : \mathfrak{l} \to \mathrm{Der}_K(K[U])$ a Lie algebra homomorphism. Choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$, and let Δ be the root system with respect to $\mathfrak{h}$. Choose a fundamental system $\Pi \subseteq \Delta$, and a corresponding Chevalley basis*

$$\{X_\gamma \mid \gamma \in \Delta\} \cup \{H_\gamma \mid \gamma \in \Pi\}.$$

Suppose that ρ satisfies

$$\exp(t\rho(X_\gamma))K[U] \subseteq K(U)(t)$$

for all $\gamma \in \Delta$, and

$$\exp(t\rho(H_\gamma))K[U] \subseteq K(U)(\exp t)$$

for all $\gamma \in \Pi$.

Then there exist an algebraic variety V containing U as an open dense subset, and a morphic action $\alpha : G \times V \to V$ of the universal connected semisimple algebraic group G with Lie algebra $\mathfrak{l}$ such that the corresponding Lie algebra homomorphism $X \mapsto -X*_\alpha$ coincides with ρ.

Here, the *universal connected semisimple algebraic group G with Lie algebra $\mathfrak{l}$* is the unique such group with the property that every finite-dimensional representation of $\mathfrak{l}$ is the differential of a representation of G.

Technical conditions on the exponentials as appearing in these theorems are shown to be necessary in Lemma 4.5.2. For example, the vector field $x^3\partial_x$ on the affine line cannot originate from an action of the additive or the multiplicative group, because $\exp(tx^3\partial_x)x = x/\sqrt{1 - 2tx^2}$, which is not a rational expression in x, t, and some exponentials $\exp(\delta_i t)$. In Example 4.5.8, Theorem 4.1.3 is applied to $\mathfrak{l} = \langle \partial_x, x\partial_x, x^2\partial_x \rangle_K$.

This chapter is organized as follows. Section 4.2 recalls standard facts on affine groups and their actions on varieties. In Section 4.3 we use these to prove existence of polynomial and rational realizations of many pairs. In Section 4.4 Theorem 4.1.1 is proved, and Section 4.5 presents the proof of our second main theorem, from which Theorems 4.1.2 and 4.1.3 readily follow. Finally, Section 4.6 discusses some possible extensions of our results.

This chapter is an expanded version of [**13**]; we thank Dmitri Zaitsev for his useful comment on an earlier version of that paper. We also thank Wilberd van der Kallen for pointing out Weil's theory of pre-transformation spaces to us, when that was exactly what we needed.

4.2. Preliminaries

We introduce some notation, and we collect some facts on affine algebraic groups that will be used later on. All of them are based on Borel's book [**5**].

Locally Finite Transformations. Let A be a finite-dimensional vector space and let $Y \in \text{End}_K(A)$. Then we write Y_s and Y_n for the semisimple part and the nilpotent part of Y, respectively. Let Γ be the $\mathbb{Z}$-module generated by the eigenvalues of Y_s on A. Decompose $A = \bigoplus_\lambda M_\lambda$, where $Y_s m = \lambda m$ for all $m \in M_\lambda$. For any $\mathbb{Z}$-module homomorphism $\phi : \Gamma \to K$, let $Y_\phi \in \text{End}_K(A)$ be defined by $Y_\phi m = \phi(\lambda)m$ for all $m \in M_\lambda$. The collection of all such Y_ϕ is denoted by $S(Y)$.

Now let A be any vector space (not necessarily finite-dimensional). A subset E of $\text{End}_K(A)$ is said to be *locally finite*, if each element of A is contained in a finite-dimensional subspace of A which is invariant under all elements of E. A representation $\rho : U \to \text{End}_K(A)$ of an associative algebra or Lie algebra U over K is called locally finite if $\rho(U)$ is locally finite. In this case, U is said to act locally finitely. A homomorphism $\rho : G \to \text{GL}(A)$ from an algebraic group G is called locally finite if $\rho(G)$ is locally finite, and in addition ρ is a homomorphism $G \to \text{GL}(M)$ of algebraic groups for each

finite-dimensional $\rho(G)$-invariant subspace M of A. In this case, G is said to act locally finitely.

If $Y \in \mathrm{End}_K(A)$ is locally finite, then the finite-dimensional Y-invariant subspaces of A form an inductive system. If $N \subseteq M \subseteq A$ are two such subspaces, then $(Y|_M)_s$ and $(Y|_M)_n$ leave N invariant, and restrict to $(Y|_N)_s$ and $(Y|_N)_n$, respectively. It follows that there are unique $Y_s, Y_n \in \mathrm{End}_K(A)$ such that $(Y_s)|_M = (Y|_M)_s$ and $(Y_n)|_M = (Y|_M)_n$ for all finite-dimensional Y-invariant subspaces M of A.

Similarly, each element of $S(Y|_M)$ leaves N invariant, and restricts to an element of $S(Y|_N)$; this restriction is surjective. Denote by $S(Y)$ the projective limit of the $S(Y|_M)$. If A has a countable basis, then $S(Y)$ projects surjectively onto each $S(Y|_M)$. Indeed, this follows from the following observation on projective limits: suppose that I is a directed set and that $(E_\alpha)_{\alpha \in I}$ is an inverse system with surjective projections $f_\alpha^\beta : E_\beta \to E_\alpha$ for all $\alpha, \beta \in I$ with $\alpha \leq \beta$. Let E be the projective limit of the E_α. In general, the projections $E \to E_\alpha$ may not all be surjective; indeed, E may be empty even if none of the E_α is ([**6**], §1, Exercise 32). However, if I contains a countable chain $\alpha_1 \leq \alpha_2 \leq \ldots$ which is *cofinal* with I, i.e., for all $\alpha \in I$ there exists an $n \in \mathbb{N}$ such that $\alpha \leq \alpha_n$, then E can be regarded as the projective limit of the E_{α_n} ([**6**], §1, no. 12) and does project surjectively onto each E_α. In our application, if $(a_n)_{n \in \mathbb{N}}$ is a basis of A, let E_n be the smallest Y-invariant subspace of A containing $a_1, \ldots, a_n$. The E_n are finite-dimensional by assumption, and form a countable chain that is cofinal with the set of all finite-dimensional Y-invariant subspaces of A. We conclude that $S(Y)$ does indeed project surjectively onto each $S(Y|_M)$.

Localization. If B is a commutative algebra, and J is an ideal in B, then we denote by $B_{\{J\}}$ the localization $B[(1 + J)^{-1}]$. If $B = K[U]$ for some irreducible affine algebraic variety U, then the elements of $B_{\{J\}}$ are rational functions on U that are defined everywhere on the zero set of J.

Comorphisms. If α is a morphism from an algebraic variety V to an algebraic variety W, and U is an open subset of W, then α induces a comorphism from the algebra of regular functions on U to the algebra of regular functions on $\alpha^{-1}(U)$. We denote this comorphism by α^0 if U is clear from the context. If W is affine, then U is implicitly assumed to be all of W. By abuse of notation, we also write α^0 for the induced comorphism of local rings $\mathcal{O}_{\alpha(p)} \to \mathcal{O}_p$, where $p \in V$, and for the comorphism $K(W) \to K(U)$ of rings of rational functions if α denotes a dominant rational map. This notation is taken from [**5**].

Differentiation of Group Actions. Let G be an affine algebraic group with unit e, V an algebraic variety, and $\alpha : G \times V \to V$ a morphic action of G on V. Then we can 'differentiate' α to a representation of $L(G) = T_e(G) = \mathrm{Der}_K(\mathcal{O}_e, K)$ as follows. Let U be an open subset of V. For $p \in U$, define the map $\alpha_p : G \to V$ by $g \mapsto \alpha(g, p)$. It maps e to p, so we may view the comorphism α_p^0 as a homomorphism $\mathcal{O}_p \to \mathcal{O}_e$. A function $f \in \mathcal{O}_V(U)$ defines an element of $\mathcal{O}_p$, to which α_p^0 may be applied. Now the function $X *_\alpha f$, defined pointwise by

$$(X *_\alpha f)(p) := (X \circ \alpha_p^0)f, \quad p \in U,$$

is an element of $\mathcal{O}_V(U)$. The map $X *_\alpha : f \mapsto X *_\alpha f$ is a K-linear derivation of $\mathcal{O}_V(U)$, and the map $X \mapsto -X *_\alpha$ is a homomorphism $L(G) \to \mathrm{Der}_K(\mathcal{O}_V(U))$ of Lie algebras.

In this way, $L(G)$ acts by derivations on the sheaf of regular functions on V. As V may not have any non-constant regular functions at all, it makes sense to compute these derivations on $\mathcal{O}_V(U)$ for an *affine* open subset U of V, so that $\mathcal{O}_V(U)$ equals the affine algebra $K[U]$. Let us assume for convenience that G and V are irreducible; then so are U and $G \times U$. In this case, α^0 sends $K[U]$ to $\mathcal{O}_{G \times V} \alpha^{-1}(U)$, an element of which defines an element of $\mathcal{O}_{G \times U}((G \times U) \cap \alpha^{-1}(U))$ by restriction. This algebra consists of fractions a/b, where $a, b \in K[G \times U]$ and b vanishes nowhere on $(G \times U) \cap \alpha^{-1}(U)$. In particular, $b(e, .)$ vanishes nowhere on U and is therefore invertible in $K[U]$. After dividing both a and b by $b(e, .)$, we have that b is an element of $1 + J$, where J is the radical ideal in $K[G \times U]$ defining $\{e\} \times U$. Thus, we can view α^0 as a map $K[U] \to K[G \times U]_{\{J\}}$. The derivation $\nabla := X \otimes I_{K[U]} : K[G] \otimes K[U] \to K[U]$ is extended to $(K[G] \otimes K[U])_{\{J\}}$ by

$$\nabla \left(\frac{a}{b} \right) = \frac{\nabla(a)b(e, .) - a(e, .)\nabla(b)}{b(e, .)^2}$$

for $a \in K[G \times U]$ and $b \in 1 + J$. As $b(e, .)$ is a non-zero constant on U, the right-hand side is an element of $K[U]$. We have thus extended $X \otimes I_{K[U]}$ to a derivation

$$(K[G] \otimes K[U])_{\{J\}} \to K[U],$$

also denoted by $X \otimes I_{K[U]}$, and we may write

(9) $$X *_\alpha = (X \otimes I_{K[U]}) \circ \alpha^0.$$

The Associative Algebra $K[G]^\vee$. The following construction is based on [**5**], §3 nr. 19. Let G be an affine algebraic group. Denote the multiplication by $\mu : G \times G \to G$, and the affine algebra by $K[G]$. For vector spaces V and W, we define a K-bilinear pairing

$$(X, Y) \mapsto X \cdot Y := (X \otimes Y) \circ \mu^0,$$
$$\operatorname{Hom}_K(K[G], V) \times \operatorname{Hom}_K(K[G], W) \to \operatorname{Hom}_K(K[G], V \otimes W).$$

The multiplication $\cdot$ turns $K[G]^\vee := \operatorname{Hom}_K(K[G], K)$ into an associative algebra, and the map

$$X \mapsto I \cdot X, \quad K[G]^\vee \to \operatorname{End}_K(K[G])$$

is a monomorphism from $K[G]^\vee$ onto the K-algebra of elements in $\operatorname{End}_K(K[G])$ commuting with all left translations λ_g for $g \in G$, which are defined by

$$(\lambda_g f)(x) = f(g^{-1}x), \quad f \in K[G].$$

We shall write $f * X$ for $(I \cdot X)f$, and $*X$ for the map $f \mapsto f * X$. In particular, $X \mapsto *X$ is a linear isomorphism from the tangent space $L(G) = T_e(G)$ onto the Lie algebra of elements of $\operatorname{Der}_K(K[G])$ commuting with all λ_g, the so-called *left-invariant vector fields*. These form a Lie algebra with respect to the commutator, and the Lie bracket on $L(G)$ is in fact defined as the pullback of that commutator under the isomorphism $X \mapsto *X$.

We recall the following well-known fact.

PROPOSITION 4.2.1. *The universal enveloping algebra $U(L(G))$ of $L(G)$ is isomorphic to the associative algebra with one generated by $L(G)$ in $K[G]^\vee$.*

Algebraicity of Lie Algebras. We reformulate some results of Chevalley on algebraicity of subalgebras of $L(G)$, where G is an affine algebraic group ([**5**], §7). For $M \subseteq L(G)$, we let $\mathcal{A}(M)$ be the intersection of all closed subgroups of G whose Lie algebras contain M, and for $X \in L(G)$ we write $\mathcal{A}(X) := \mathcal{A}(\{X\})$. A subalgebra $\mathfrak{l}$ of $L(G)$ is called *algebraic* if $L(\mathcal{A}(\mathfrak{l})) = \mathfrak{l}$; an element X of $L(G)$ is called algebraic if $\langle X \rangle_K$ is an algebraic subalgebra of $L(G)$.

Recall that $*(K[G]^\vee)$ is locally finite (compare the proof of Proposition 4.4.1). For $X \in L(G)$, both the semisimple part and the nilpotent part of $*X$ are in $*L(G)$, and we denote their pre-images in $L(G)$ by X_s and X_n, respectively. As $K[G]$ contains a finite-dimensional faithful $L(G)$-module that generates $K[G]$ as an algebra, the $\mathbb{Z}$-module Γ_X of eigenvalues of X_s in $K[G]$ is finitely generated. As Γ_X is a torsion-free Abelian group, it is free, and we may choose a basis $\lambda_1, \ldots, \lambda_d$ of Γ_X. For a variable t, consider the map $\exp(tX) : K[G] \to K[[t]]$ defined by

$$\exp(tX)f = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n(f), \quad f \in K[G]$$

where X^n is viewed as an element of $K[G]^\vee$. Alternatively, we could write this formal power series as $(\exp(t(*X))f)(e)$, where the exponential is the one defined in Section 4.1. Clearly, $\exp(tX)$ is a homomorphism of K-algebras, whence an element of $G(K[[t]])$ ([**5**], §1 nr. 5). However, the following lemma shows that the image lies in a much smaller algebra.

LEMMA 4.2.2. *If $X_n = 0$, then the map $\exp(tX)$ is a homomorphism*

$$K[G] \to K[\exp(\pm\lambda_1 t), \ldots, \exp(\pm\lambda_d t)],$$

whence the comorphism of a homomorphism $\gamma : (K^*)^d \to G$ of algebraic groups. If $X_n \neq 0$, then $\exp(tX)$ is a homomorphism

$$K[G] \to K[t, \exp(\pm\lambda_1 t), \ldots, \exp(\pm\lambda_d t)],$$

whence the comorphism of a homomorphism $\gamma : K \times (K^*)^d \to G$ of algebraic groups. In either case, γ is an algebraic group monomorphism onto $\mathcal{A}(X)$.

PROOF. By [**5**], §1 nr. 11 we may assume that G is a closed subgroup of GL_n for some n, and we may view X as an element of $\mathfrak{gl}_n$. After a change of basis, $X_s = \mathrm{diag}(\nu_1, \ldots, \nu_n)$, where $\nu_1, \ldots, \nu_n$ generate Γ_X; it follows that

$$\exp(tX_s) = \mathrm{diag}(\exp(\nu_1 t), \ldots, \exp(\nu_n t)).$$

We have

$$\exp(\nu_1 t)^{m_1} \cdots \exp(\nu_n t)^{m_n} = 1$$

for any $\mathbf{m} \in \mathbb{Z}^n$ such that $\sum_i m_i \nu_i = 0$; hence by [**5**], §7 nr. 3 we have

$$\exp tX_s \in \mathcal{A}(X_s)(K[\exp(\pm\nu_1 t), \ldots, \exp(\pm\nu_n t)]).$$

Now, any specialization

$$K[\exp(\pm\nu_1 t), \ldots, \exp(\pm\nu_n t)] \to K$$

sends $\exp(tX_s)$ to an element of $\mathcal{A}(X_s)$. The algebra on the left-hand side is isomorphic to the algebra

$$K[\exp(\pm\lambda_1 t), \ldots, \exp(\pm\lambda_d t)],$$

and as the λ_i are linearly independent over $\mathbb{Q}$, the latter is the affine algebra of $(K^*)^d$. We have thus constructed the algebraic group homomorphism $G_m^d \to \mathcal{A}(X_s)$. It is injective, as the ν_i generate the same $\mathbb{Z}$-module as the λ_i. As d is also the dimension of $\mathcal{A}(X_s)$, and as $\mathcal{A}(X_s)$ is connected, the homomorphism is surjective onto $\mathcal{A}(X_s)$.

In [**5**], §7 nr. 3 it is also proved that the homomorphism $G_a \to G$ corresponding to the comorphism $\exp(tX_n) : K[G] \to K[t]$ is a monomorphism from G_a onto $\mathcal{A}(X_n)$. The lemma now follows from $\exp(tX) = \exp(tX_n)\exp(tX_s)$ and $\mathcal{A}(X) = \mathcal{A}(X_n) \times \mathcal{A}(X_s)$ (direct product). $\square$

Recall the notation $S(\cdot)$ of page 57. For $X \in L(G)$ the set $S(*X)$ is a subset of $*L(G)$; we denote its pre-image in $L(G)$ by $S(X)$. Now $L(\mathcal{A}(X))$ is spanned by X_n and $S(X)$. More generally, we have the following theorem of Chevalley ([**5**], §7 nrs. 3 and 7).

THEOREM 4.2.3. *Let M be a subset of $L(G)$. Then $L(\mathcal{A}(M))$ is generated by the X_n and $S(X)$ as X varies over M.*

Example 4.4.3 shows a subalgebra $\mathfrak{l} \subseteq L(G)$ that is not equal to $L(\mathcal{A}(\mathfrak{l}))$. An algebraic element X of $L(G)$ is either nilpotent with $\Gamma_X = \{0\}$ or semisimple with Γ_X of rank 1. Accordingly, $\mathcal{A}(X)$ is isomorphic to G_a or G_m, and if we denote the usual affine coordinate on the additive or multiplicative group by Y, then the differential at the identity of the homomorphism $G_a \to G$ (respectively $G_m \to G$) constructed above sends the basis vector $\partial_Y|_0$ of $L(G_a)$ (respectively $Y\partial_Y|_1$ of $L(G_m)$) to X.

4.3. Polynomial Realizations

The following proposition is immediate from the differentiation of group actions of page 58.

PROPOSITION 4.3.1. *Let G be an affine algebraic group, and let H be a closed subgroup of G. Suppose that G/H has a smooth G-equivariant completion, in which eH has an open neighbourhood isomorphic to an affine space. Then $(L(G), L(H))$ has a polynomial transitive realization.*

COROLLARY 4.3.2. *Let G be a connected reductive algebraic group, and let P a parabolic subgroup of G. Then $(L(G), L(P))$ has a polynomial transitive realization.*

PROOF. By the Bruhat decomposition, G/P has a partition into cells, i.e., locally closed subsets that are isomorphic to affine spaces. Among them is a unique cell of dimension $\dim G/P$: the big cell. If necessary, the action of G can move the big cell such that eP is contained in it. Now apply Proposition 4.3.1 to the complete variety G/P itself. $\square$

Michel Brion pointed out the following stronger corollary to me.

COROLLARY 4.3.3. *Let G be a connected reductive algebraic group, and let H be a closed subgroup of G such that some Borel subgroup has a dense orbit on G/H. Then $(L(G), L(H))$ has a polynomial transitive realization.*

Such a subgroup H of G is called *spherical*, and the corresponding homogeneous space G/H is called a *spherical variety*. Any such variety has a smooth equivariant completion in which eH has an open neighbourhood isomorphic to an affine space [**10**], [**11**], whence the corollary.

COROLLARY 4.3.4. *Let $\mathfrak{k}$ be a simple Lie algebra, and let $\mathfrak{d}$ be the diagonal subalgebra of $\mathfrak{k} \oplus \mathfrak{k}$. Then $(\mathfrak{k} \oplus \mathfrak{k}, \mathfrak{d})$ has a polynomial transitive realization.*

PROOF. Let K be any of the connected algebraic groups with Lie algebra $\mathfrak{k}$, and let D be the diagonal subgroup of $G := K \times K$. Then D is easily shown to be a symmetric subgroup, and Corollary 4.3.3 applies.   □

After having proved the existence of realizations of diagonal pairs with coefficients in E (Theorem 2.5.3), and having shown that most such pairs cannot be embedded into simple-parabolic ones (Proposition 3.5.3), the above corollary finally settles the realization problem for diagonal pairs—at least in theory. It would be interesting to see if one can actually compute such polynomial realizations.

4.4. The Locally Finite Case

In this section we prove Theorem 4.1.1. Let G be an affine algebraic group, U an affine algebraic variety, and $\alpha : G \times U \to U$ a morphic action. Then the Lie algebra homomorphism $X \mapsto -X *_\alpha$ of page 58 can be described more directly.

PROPOSITION 4.4.1. *For $X \in K[G]^\vee$, define the K-linear map $X *_\alpha : K[U] \to K[U]$ by*

(10)
$$X *_\alpha := (X \otimes I_{K[U]}) \circ \alpha^0.$$

Then $X \mapsto X *_\alpha$ is an anti-homomorphism of associative K-algebras. Moreover, the algebra $(K[G]^\vee) *_\alpha$ is locally finite.

PROOF. The fact that α is an action can be expressed in terms of comorphisms by
$$(\mu^0 \otimes I_{K[U]}) \circ \alpha^0 = (I_{K[G]} \otimes \alpha^0) \circ \alpha^0.$$
Let $X, Y \in K[G]^\vee$, and compute
$$(X \cdot Y) *_\alpha = (((X \otimes Y) \circ \mu^0) \otimes I_{K[U]}) \circ \alpha^0$$
$$= (X \otimes Y \otimes I_{K[U]}) \circ (\mu^0 \otimes I_{K[U]}) \circ \alpha^0,$$
which, by the above remark, equals
$$(X \otimes Y \otimes I_{K[U]}) \circ (I_{K[G]} \otimes \alpha^0) \circ \alpha^0$$
$$= (Y \otimes I_{K[U]}) \circ \alpha^0 \circ (X \otimes I_{K[U]}) \circ \alpha^0$$
$$= (Y *_\alpha) \circ (X *_\alpha).$$

This proves the first statement. Next, if $f \in K[U]$ and $\alpha^0(f) = \sum_{i=1}^k a_i \otimes b_i$ with $a_i \in K[G]$ and $b_i \in K[U]$, then clearly $(K[G]^\vee) *_\alpha f \subseteq \langle b_1, \ldots, b_k \rangle_K$. This proves the second statement.   □

Note that the local finiteness of $(K[G]^\vee)*_\mu$ on $K[G]$ is a special case of this proposition. The proof that $*(K[G]^\vee)$ is locally finite, a fact that we used in formulating Chevalley's results on page 60, is very similar. Also note the subtle difference between the seemingly identical formulas (9) and (10). In the latter, α^0 is a map $K[U] \to K[G \times U]$, whereas in the former, it is a map $K[U] \to K[G \times U]_{\{J\}}$.

As a consequence of Proposition 4.4.1, the representation $X \mapsto -X*_\alpha$ of $L(G)$ on $K[U]$ is locally finite, and so is the representation $G \to \mathrm{Aut}(K[U])$ defined by $g \mapsto \lambda_g$, where $(\lambda_g f)(p) := f(g^{-1}p)$. The latter follows because $\lambda_g = e_{g^{-1}}*_\alpha$, where $e_g \in K[G]^\vee$ denotes evaluation in g. In fact, $X \mapsto -X*_\alpha$ is the derivative at e of the map $g \mapsto \lambda_g$. Conversely, we have the following theorem.

THEOREM 4.4.2. *Let B be a finitely generated K-algebra (not necessarily commutative), and let $\mathfrak{l} \subseteq \mathrm{Der}_K(B)$ be a finite-dimensional Lie subalgebra acting locally finitely on B. Then there exist an affine algebraic group G, a faithful locally finite representation $\rho : G \to \mathrm{Aut}(B)$, and an embedding $\phi : \mathfrak{l} \to L(G)$ such that $(d_e\rho) \circ \phi = \mathrm{id}$.*

PROOF. For any finite-dimensional $\mathfrak{l}$-invariant subspace M of B, denote by $\mathfrak{l}_M$ the restriction of $\mathfrak{l}$ to M, and set $\tilde{\mathfrak{l}}_M := L(\mathcal{A}(\mathfrak{l}_M))$; here $\mathcal{A}$ is defined with respect to the algebraic group $\mathrm{GL}(M)$, whose Lie algebra is naturally identified with $\mathfrak{gl}(M)$, which contains $\mathfrak{l}_M$. The $\tilde{\mathfrak{l}}_M$ form an inverse system, whose projections are surjective by Theorem 4.2.3 and the projection properties of X_n and $S(X)$ (page 57). Let $\tilde{\mathfrak{l}}$ be the projective limit of this system. As B is finitely generated, its dimension is (at most) countable, and the projections $\tilde{\mathfrak{l}} \to \tilde{\mathfrak{l}}_M$ are all surjective; see page 57.

By Theorem 4.2.3, the space $\tilde{\mathfrak{l}}$ can be viewed as the Lie subalgebra of $\mathrm{End}_K(B)$ generated by the X_n and $S(X)$ as X varies over $\mathfrak{l}$. We claim that these are all derivations of B. To verify this, it suffices to check Leibniz' rule on eigenvectors of X_s. To this end, let $a, b \in B$ be such that $X_s a = \lambda a$ and $X_s b = \mu b$. This is equivalent to

$$(X - \lambda)^k a = (X - \mu)^l b = 0$$

for some $k, l \in \mathbb{N}$. From the identity

$$(X - (\lambda + \mu))^m (ab) = \sum_{i=0}^{m} \binom{m}{i}(X - \lambda)^i(a)(X - \mu)^{m-i}(b)$$

it follows that the left-hand side is zero for some $m \in \mathbb{N}$. Hence,

$$X_s(ab) = (\lambda + \mu)ab = X_s(a)b + aX_s(b),$$

and X_s is a derivation, and so is $X_n = X - X_s$. Now let ϕ be a $\mathbb{Z}$-module homomorphism from the $\mathbb{Z}$-span of the eigenvalues of X_s to K. Then the map $X_\phi \in S(X)$ satisfies

$$(X_\phi a)b + a(X_\phi b) = (\phi(\lambda) + \phi(\mu))ab = \phi(\lambda + \mu)ab = X_\phi(ab).$$

We have thus found that $\tilde{\mathfrak{l}}$ is generated by, and hence consists of, derivations. Let M be a finite-dimensional $\mathfrak{l}$-submodule of B that generates B as an algebra. We have seen that the projection $\tilde{\mathfrak{l}} \to \tilde{\mathfrak{l}}_M$ is surjective, but as $\tilde{\mathfrak{l}}$ consists of derivations, which are determined by their values on M, it is also injective. Hence, $\tilde{\mathfrak{l}}_M$ acts on B by derivations. Let $G \subseteq \mathrm{GL}(M)$ be $\mathcal{A}(\mathfrak{l}_M)$. It follows that G acts locally finitely, and by automorphisms, on B; by construction, the corresponding action of $L(G) = \tilde{\mathfrak{l}}_M$ restricts to the identity on $\mathfrak{l}$. $\qquad\square$

Note that the construction of G does not depend on the choice of M. The triple (G, ρ, ϕ) constructed in the proof has the property that $\mathcal{A}(\phi(\mathfrak{l})) = G$, and with this additional condition it is unique in the following sense: if (G', ρ', ϕ') is another such triple, then there exists an isomorphism $\psi : G \to G'$ such that $\rho' \circ \psi = \rho$ and $(d_e\psi) \circ \phi = \phi'$. Indeed, for any finite-dimensional G'-invariant subspace M of B that generates B as an algebra, G' must be isomorphic to $\mathcal{A}(\mathfrak{l}_M)$, just like G; this defines the required isomorphism ψ.

Now the first main theorem follows almost directly.

PROOF OF THEOREM 4.1.1. Apply Theorem 4.4.2 to $B = K[U]$ to find G and its representation on $K[U]$. Let $M \subseteq K[U]$ be a finite-dimensional G-invariant subspace that generates $K[U]$ as an algebra. Then the surjective G-equivariant map from the symmetric algebra generated by M onto $K[U]$ allows us to view U as a closed G-invariant subset of the dual $M^\vee$. This gives the morphic action of G on U, and it is straightforward to verify the required property. $\square$

Let us consider two examples where the embedding $\mathfrak{l} \to L(G)$ is not an isomorphism.

EXAMPLE 4.4.3. Let $U := \operatorname{Spec}_K K[x, y]$ be the affine plane, and let

$$\mathfrak{l} := \langle \lambda_1 x\partial_x + \lambda_2 y\partial_y, \partial_y, x\partial_y, \dots, x^r\partial_y \rangle_K,$$

where $\lambda_1, \lambda_2 \in K$ are linearly independent over $\mathbb{Q}$, and $r \in \mathbb{N}$. The Lie algebra $\mathfrak{l}$ acts locally finitely on $K[x, y]$. Indeed, for $f \in K[x, y]$, any element $g \in U(\mathfrak{l})f$ satisfies

$$\deg_x(g) \leq \deg_x(f) + r \deg_y(f), \text{ and } \deg_y(g) \leq \deg_y(f).$$

Hence, Theorem 4.4.2 applies. Following its proof, we choose the $\mathfrak{l}$-invariant space $M = \langle y, 1, x, x^2, \dots, x^r \rangle_K$, which generates $K[x, y]$. Denoting by $\mathfrak{l}_M$ the restriction of $\mathfrak{l}$ to M, the proof of Theorem 4.1.1 shows that $\tilde{\mathfrak{l}}_M := L(\mathcal{A}(\mathfrak{l}_M))$ acts by derivations on $K[x, y]$. With respect to the given basis of M, the derivation $\lambda_1 x\partial_x + \lambda_2 y\partial_y$ has matrix

$$\operatorname{diag}(\lambda_2, 0, \lambda_1, 2\lambda_1, \dots, r\lambda_1),$$

whereas the elements $x^i\partial_y$ of $\mathfrak{l}$ act nilpotently on M. Hence, $\tilde{\mathfrak{l}}_M$ is generated (and in fact spanned) by $\mathfrak{l}_M$ and the linear map with matrix

$$\operatorname{diag}(1, 0, \dots, 0).$$

The image of $\tilde{\mathfrak{l}}_M$ in $\operatorname{Der}_K(K[x, y])$ is spanned by $\mathfrak{l}$ and $x\partial_x$. The algebraic group G is a semi-direct product $G_m^2 \ltimes G_a^{r+1}$ acting by

$$(t_1, t_2, a_0, \dots, a_r)(x, y) := (t_1 x, t_2 y + \sum_{i=0}^r a_i x^i).$$

EXAMPLE 4.4.4. Let $U := \operatorname{Spec}_K(K[x, y])$ be the affine plane, and let $\mathfrak{l}$ be the one-dimensional Lie algebra spanned by $\partial_x + y\partial_y$. Clearly, $\mathfrak{l}$ acts locally finitely on $K[x, y]$; the group G of Theorem 4.1.1 is $G_a \times G_m$ acting on U by $(a, b)(x, y) = (x + a, by)$, and the image of its Lie algebra in $\operatorname{Der}_K(K[U])$ equals $\langle \partial_x, y\partial_y \rangle$.

4.5. The General Case

This section is concerned with Theorem 4.5.4. The need for this theorem becomes clear from the following example.

EXAMPLE 4.5.1. Let SL_{n+1} act on the projective n-space in the natural way. Then, after choosing suitable coordinates x_i on an affine part $\mathbb{A}^n \subseteq \mathbb{P}^n$, the corresponding homomorphism $X \mapsto -X*_\alpha$, $\mathfrak{sl}_{n+1} \to \mathrm{Der}_K(K[\mathbb{A}^n])$ has image

$$\langle \{\partial_i, x_i\partial_j, x_iE\}_{i,j}\rangle_K,$$

where $E = \sum_i x_i\partial_i$. Clearly, this Lie algebra is not locally finite, so that we cannot apply Theorem 4.1.1.

Let G be a connected affine algebraic group acting on a an irreducible algebraic variety V by means of a morphic action $\alpha : G \times V \to V$, and let $U \subseteq V$ be an open affine subvariety. Recall the definition of the map $X \mapsto X*_\alpha$, $L(G) \to \mathrm{Der}_K(K[U])$ and the definition of the exponential map from Section 4.1 as well as the definition of Γ_X of page 60.

LEMMA 4.5.2. *Let $X \in L(G)$ and let $\lambda_1, \ldots, \lambda_d$ be a basis for Γ_X. Then*

$$\exp(t(X*_\alpha))K[U] \subseteq K[U][t, s_1, \ldots, s_d]_{\{P\}}$$

where $s_i = \exp(\lambda_i t)$, and P is the ideal generated by $t, s_1 - 1, \ldots, s_d - 1$. In particular, if X is algebraic, then

$$\exp(t(X*_\alpha))K[U] \subseteq K[U][t]_{\{(t)\}}$$

if X is nilpotent, and

$$\exp(t(X*_\alpha))K[U] \subseteq K[U][s_1]_{\{(s_1-1)\}}$$

if X is semisimple.

PROOF. We claim that

(11) $$(X*_\alpha)^n = (X^n \otimes I_{K[U]}) \circ \alpha^0.$$

where X^n is evaluated in the associative algebra $K[G]^\vee$. To prove this, proceed by induction on n. For $n = 1$ it is Equation (9); suppose that it holds for n, and compute

$$
\begin{aligned}
(X*_\alpha)^{n+1} &= (X \otimes I_{K[U]}) \circ \alpha^0 \circ (X^n \otimes I_{K[U]}) \circ \alpha^0 \\
&= (X \otimes I_{K[U]}) \circ (X^n \otimes I_{K[G]} \otimes I_{K[U]}) \circ (I_{K[G]} \otimes \alpha^0) \circ \alpha^0 \\
&= (X \otimes I_{K[U]}) \circ (X^n \otimes I_{K[G]} \otimes I_{K[U]}) \circ (\mu^0 \otimes I_{K[U]}) \circ \alpha^0 \\
&= (((X^n \otimes X) \circ \mu^0) \otimes I_{K[U]}) \circ \alpha^0 \\
&= (X^{n+1} \otimes I_{K[U]}) \circ \alpha^0.
\end{aligned}
$$

In the first equality, we used the induction hypothesis, and in the third we used the fact that α is a morphic action. The last equality uses the multiplication in $K[G]^\vee$ as defined in Section 4.2. The other equalities follow from easy tensor product manipulations.

Using (11), we calculate

$$
\begin{aligned}
\exp(tX*_\alpha) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}(X^n \otimes I_{K[U]}) \circ \alpha^0 \\
&= (\exp(tX) \otimes I_{K[U]}) \circ \alpha^0.
\end{aligned}
$$

Now α^0 is a map $K[U] \to (K[G] \otimes K[U])_{\{J\}}$, where J is as in Section 4.1, and $\exp(tX)$ maps $K[G]$ into $K[t, s_1^{\pm 1}, \ldots, s_d^{\pm 1}]$. Under $\exp(tX)$, the ideal J is mapped into the ideal P. This concludes the proof.   □

REMARK 4.5.3. The proof of Lemma 4.5.2 shows that $\exp(tX *_\alpha)$ can be viewed as the comorphism of the rational map $\mathcal{A}(X) \times U \to U$ defined by the restriction of α.

Now suppose that we are given a homomorphism $L(G) \to \mathrm{Der}_K(K[U])$ for some affine algebraic variety U. Then the above lemma gives a necessary condition for this homomorphism to come from a group action on an algebraic variety V containing U as an open subset. In a sense, this condition is also sufficient, as the following theorem shows. To appreciate its formulation, recall from page 61 that if X is algebraic, then Γ_X has rank zero or one according to whether X is nilpotent or semisimple.

THEOREM 4.5.4. *Let G be a connected affine algebraic group and let $X_1, \ldots, X_k$ be a basis of $L(G)$ consisting of algebraic elements. Let U be an irreducible affine algebraic variety, and let $\rho : L(G) \to \mathrm{Der}_K(K[U])$ be a homomorphism of Lie algebras.*

Denote by Σ the set of indices i for which X_i is semisimple (in its action on $K[G]$), and by N the set of indices i for which X_i is nilpotent. For $i \in \Sigma$, let $\lambda_i \in K$ be a generator of Γ_{X_i}.

Assume that the product map

$$\pi : \mathcal{A}(X_1) \times \ldots \times \mathcal{A}(X_k) \to G$$

maps an open neighbourhood of $(e, \ldots, e)$ isomorphically onto an open neighbourhood of $e \in G$, and suppose that

$$\exp(t\rho(X_i)) \in \begin{cases} K(U)(t) & \text{if } i \in N, \text{ and} \\ K(U)(\exp(\lambda_i t)) & \text{if } i \in \Sigma. \end{cases}$$

Then there exist an algebraic variety V containing U as an open dense subset, and a morphic action $\alpha : G \times V \to V$, such that the map $X \mapsto -X *_\alpha$, $L(G) \to \mathrm{Der}_K(K[U])$ coincides with ρ. Indeed, up to equivalence, there exists a unique such pair (V, α) with the additional property that $V \setminus U$ contains no G-orbit.

Before proving this theorem, let us recall Weil's results on pre-transformation spaces ([**60**], or Zaitsev's paper [**63**] for a generalization to non-irreducible varieties); rather than stating these in full generality, we shall adjust the formulation to our specific needs. Recall that a rational map β has a natural 'largest possible' domain; this domain is denoted by $\mathrm{dom}(\beta)$.

LEMMA 4.5.5. *Let G be a connected algebraic group with multiplication $\mu : G \times G \to G$, U an algebraic variety, and $\beta : G \times U \to U$ a dominant rational map such that*

$$\beta \circ (\mathrm{id}_G \times \beta) = \beta \circ (\mu \times \mathrm{id}_U)$$

as dominant rational maps $G \times G \times U \to U$. Assume, moreover, that $\{e\} \times U \subseteq \mathrm{dom}(\beta)$, and that $\beta(e, p) = p$ for all $p \in U$.

Then there exist an algebraic variety V, an open immersion $\psi : U \to V$ with dense image, and a morphic action $\alpha : G \times V \to V$ such that $\alpha \circ (\mathrm{id}_G \times \psi)$ and β define the same rational map. Indeed, up to equivalence, there exists a unique such triple (V, ψ, α) with the additional property that $V \setminus \psi(U)$ contains no G-orbit.

Zaitsev calls the pair (U, β) a *pre-transformation G-space*, and the triple (V, ψ, α) a *regularization* of (U, β); equivalence of these is defined in the obvious manner. If $V \setminus \psi(U)$ contains no G-orbit, the regularization is called *minimal*, as in that case no proper open subset of V is also a regularization of (U, α).

PROOF. We show that β makes U into a 'pre-transformation G-space', in which every point is a 'point of regularity' in the sense of [**63**]. First, 'generic associativity' follows from the condition on β. Second, we must show 'generic existence and uniqueness of left divisions', i.e., that the rational map $(g, p) \mapsto (g, \beta(g, p))$ is in fact a birational map $G \times U \to G \times U$. Indeed, using generic associativity and the fact that $\beta(e, p) = p$ for all $p \in U$, we find that the rational map $(g, p) \mapsto (g, \beta(g^{-1}, p))$ is inverse to it.

Finally, let $p_0 \in U$. Then the set Ω of $g \in G$ for which both $(g, p_0) \in \mathrm{dom}(\beta)$ and $(g^{-1}, (\beta(g, p_0))) \in \mathrm{dom}(\beta)$ is open, and non-empty as $e \in \Omega$. As G is connected, Ω is dense in G. Let $g_0 \in \Omega$, and consider the following rational maps $U \to U$: $p \mapsto \beta(g_0, p)$ and $p \mapsto \beta(g_0^{-1}, p)$. The first is defined at p_0 and the second at $\beta(g_0, p_0)$. Hence, both compositions are rational maps $U \to U$. Again, using generic associativity and the fact that $\beta(e, p) = p$ for all $p \in U$, we find that the two maps are each other's inverses. This shows that p_0 is a point of regularity. We may now apply Theorem 4.11 of [**63**] to find (α, V). The proof of this theorem shows that α, which is *a priori* just a birational map $U \to V$, is an open immersion on the set of points of regularity, which is all of U. Finally, the remark just before Theorem 4.9 of [**63**] shows that α is a morphic group action of G on V. □

PROOF OF THEOREM 4.5.4. Let $\epsilon_i = 0$ or 1 if $i \in N$ or $i \in \Sigma$, respectively. By the property of π, the homomorphism

$$\exp(t_1 X_1) \cdot \ldots \cdot \exp(t_k X_k)$$

identifies $K(G)$ with the field $K(s_1, \ldots, s_k)$, and $\mathcal{O}_e$ with the localization $K[\mathbf{s}]_M := K[s_1, \ldots, s_k]_M$, where $s_i = t_i$ if $i \in N$ and $s_i = \exp(\lambda_i t_i)$ if $i \in \Sigma$, and M is the (maximal) ideal generated by the elements $s_i - \epsilon_i$. The K-algebra $K[[\mathbf{t}]] := K[[t_1, \ldots, t_k]]$ can now be viewed as $\hat{\mathcal{O}}_e$, the completion of $K[\mathbf{s}]_M$ with respect to the M-adic topology.

The co-multiplication

$$\mu^0 : \mathcal{O}_e = K[\mathbf{s}]_M \to K[\mathbf{s}', \mathbf{s}'']_{M' \otimes K[\mathbf{s}''] + K[\mathbf{s}'] \otimes M''} = \mathcal{O}_{(e,e)}$$

extends uniquely to a continuous homomorphism

$$\mu^0 : \hat{\mathcal{O}}_e = K[[\mathbf{t}]] \to K[[\mathbf{t}', \mathbf{t}'']] = \hat{\mathcal{O}}_{(e,e)},$$

and we have

(12) $\quad \exp(\mu^0(t_1)X_1) \cdots \exp(\mu^0(t_k)X_l)$
$$= \exp(t_1' X_1) \cdots \exp(t_k' X_k) \cdot \exp(t_1'' X_1) \cdots \exp(t_k'' X_k).$$

Similarly, the evaluation map $f \mapsto f(e)$, $\mathcal{O}_e \to K$ extends to the continuous map $f \mapsto f(0)$, $K[[\mathbf{t}]] \to K$, where K is given the discrete topology. Also, $X_i \in \mathrm{Der}_K(\mathcal{O}_e, K)$ extends uniquely to a continuous K-linear derivation $K[[\mathbf{t}]] \to K$, where K is given the structure of a $K[[\mathbf{t}]]$-module defined by $fc = f(0)c$. This extension satisfies

(13) $$X_i(t_j) = \delta_{i,j}.$$

Consider the map

$$\beta^0 := \exp(-t_k\rho(X_k))\cdots\exp(-t_1\rho(X_1)) : K[U] \to K[U][[t_1,\ldots,t_k]],$$

where we implicitly extend each $\rho(X_i)$ linearly and continuously to formal power series with coefficients from $K[U]$. From the fact that the $\rho(X_i)$ are derivations, one finds that β^0 is a homomorphism. It is clearly injective, hence it extends to an injective homomorphism $K(U) \to K(U)(s_1,\ldots,s_k)$ by assumption. The latter field is identified with $K(G \times U)$ by the identification of $K(G)$ with $K(s_1,\ldots,s_k)$, and it follows that we may view β^0 as the comorphism of a dominant rational map $\beta : G \times U \to U$. We claim that the triple (G, U, β) satisfies the conditions of Lemma 4.5.5.

Denote by P the ideal in $K[U][s_1,\ldots,s_k]$ generated by the $s_i - \epsilon_i$, and let $f \in K[U]$. As $\beta^0(f)$ is both a power series in the t_i and an element of $K(U)(s_1,\ldots,s_k)$, we have

$$\beta^0(f) \in K[U][s_1,\ldots,s_k]_{\{P\}}$$

in the notation of page 58. We may identify the algebra on the right-hand side with the algebra $K[G \times U]_{\{J\}}$, where J is the radical ideal in $K[G \times U]$ defining $\{e\} \times U$. Hence, $\{e\} \times U \subseteq \mathrm{dom}(\beta^0(f))$ for all $f \in K[U]$, which proves that $\{e\} \times U \subseteq \mathrm{dom}(\beta)$. Moreover, $\beta^0(f)(e,p) = f(p)$ for all $f \in K[U]$, from which it follows $\beta(e,p) = p$ for all $p \in U$.

Before proving generic associativity, we extend the map $-\rho$ to an anti-homomorphism τ from $U(L(G))$ into $\mathrm{End}_K(K[U])$, which can be done in a unique way. By Proposition 4.2.1 we may view $U(L(G))$ as the associative algebra with one generated by $L(G)$ in $K[G]^\vee$. We extend τ linearly and continuously to formal power series with coefficients from $U(L(G))$. Also, we extend the map $\mu^0 : K[[t_1,\ldots t_k]] \to K[[t_1',\ldots,t_k',t_1'',\ldots,t_k'']]$ to a map

$$U(L(G))[[t_1,\ldots,t_k]] \to U(L(G))[[t_1',\ldots,t_k',t_1'',\ldots,t_k'']]$$

by

$$\mu^0(\sum_{m\in\mathbb{N}^k} u_m t^m) = \sum_{m\in\mathbb{N}^k} u_m\mu^0(t^m).$$

Note that $\tau \circ \mu^0 = \mu^0 \circ \tau$; indeed, τ acts only on $U(L(G))$ and μ^0 only on the t_i.

We want to prove that

$$\beta \circ (\mathrm{id}_G \times\beta) = \beta \circ (\mu \times \mathrm{id}_U)$$

as dominant rational maps $G \times G \times U \to U$. This is equivalent to

$$(I_{K[G]} \otimes \beta^0) \circ \beta^0 = (\mu^0 \otimes I_{K[U]}) \circ \beta^0,$$

for the comorphisms $K(U) \to K(G \times G \times U)$, and it suffices to prove this for the corresponding homomorphisms

$$K[U] \to K[U][[t_1',\ldots,t_k',t_1'',\ldots,t_k'']],$$

where we use $t_1',\ldots,t_k'$ for the generators of $\hat{\mathcal{O}}_e$ on the first copy of G, and $t_1'',\ldots,t_k''$ for those on the second copy. Compute

$$\begin{aligned}
&(I_{K[G]} \otimes \beta^0) \circ \beta^0 \\
&= \exp(-t_k''\rho(X_k))\cdots\exp(-t_1''\rho(X_1)) \cdot \exp(-t_k'\rho(X_k))\cdots\exp(-t_1'\rho(X_1)) \\
&= \tau(\exp(t_1'X_1)\cdots\exp(t_k'X_k)\exp(t_1''X_1)\cdots\exp(t_k''X_k)),
\end{aligned}$$

to which we apply Equation (12), and find

$$
\begin{aligned}
&\tau(\exp(\mu^0(t_1)X_1)\cdots\exp(\mu^0(t_k)X_k))\\
&= \tau(\mu^0(\exp(t_1X_1)\cdots\exp(t_kX_k)))\\
&= \mu^0(\tau(\exp(t_1X_1)\cdots\exp(t_kX_k)))\\
&= \mu^0(\exp(-t_k\rho(X_k))\cdots\exp(-t_1\rho(X_1)))\\
&= (\mu^0\otimes I_{K[U]})\circ\beta^0,
\end{aligned}
$$

as required.

Now that we have checked the conditions of Lemma 4.5.5, let V and $\alpha : G\times V\to V$ be as in the conclusion of that lemma. For $f\in K[U]$ we have

$$
\begin{aligned}
-(X_i *_\alpha f) &= (-X_i\otimes I)\alpha^0(f)\\
&= (-X_i\otimes I)\beta^0(f)\\
&= (-X_i\otimes I)(\exp(-t_k\rho(X_k))\cdots\exp(-t_1\rho(X_1)))f)\\
&= \rho(X_i)(f).
\end{aligned}
$$

In the last step we used that $X_i(t_j) = \delta_{i,j}$. This finishes the proof of the existence of V and α.

As for the uniqueness, suppose that V and α satisfy the conclusions of the theorem. Then α defines a rational map $G\times U\to U$. From Remark 4.5.3, we find that this rational map coincides with β defined above. Hence, the uniqueness of (V,α) follows from the uniqueness of a minimal regularization of (U,β); see Lemma 4.5.5. $\qquad\square$

The following lemma shows that the conditions on G in Theorem 4.5.4 are not all that rare.

LEMMA 4.5.6. *Let G be a connected affine algebraic group over K. Then G has one-dimensional closed connected subgroups $H_1,\ldots,H_k$ such that the product map $H_1\times\ldots\times H_k\to G$ is an open immersion.*

This fact is well known; see for example [**29**] and [**5**]. As we use the proof later, e.g. in the proof of Theorem 4.1.3, we give a brief sketch of the proof.

PROOF. By a result of Mostow, the unipotent radical $R_u(G)$ of G has a reductive Levi complement G', i.e., $G = G'\ltimes R_u(G)$ ([**5**], §11 nr. 22). Hence, it suffices to prove the proposition for G reductive and for G unipotent.

If G is unipotent, then there exists a basis $X_1,\ldots,X_k$ of $L(G)$ such that $\langle X_i,\ldots,X_k\rangle$ is an ideal in $L(G)$ for all i, and the $H_i = \mathcal{A}(X_i)\cong G_a$ are subgroups as required.

If G is reductive, choose a maximal torus $T\subseteq G$, and a Borel subgroup $B^+\subseteq G$ containing T. Let B^- be the opposite Borel subgroup ([**5**], §14 nr. 1), and set $U^{\pm} := R_u(B^{\pm})$. Then it is known that the product map $U^-\times T\times U^+\to G$ is an open immersion; now U^- and U^+ are dealt with by the unipotent case, and T is isomorphic to G_m^d for some d. $\qquad\square$

REMARK 4.5.7. It is not true that the product map $\mathcal{A}(X_1)\times\ldots\times\mathcal{A}(X_k)\to G$ is an open immersion for every basis $X_1,\ldots,X_k$ of $L(G)$ consisting of algebraic elements.

Indeed, consider $G = G_m^2$ with Abelian Lie algebra K^2. The elements $X_1 = (1,0), X_2 = (1,2)$ are algebraic, and form a basis of $L(G)$. We have

$$\mathcal{A}(X_1) = \{(a,1) \mid a \in K^*\}, \text{ and } \mathcal{A}(X_2) = \{(b,b^2) \mid b \in K^*\}.$$

The product map $\mathcal{A}(X_1) \times \mathcal{A}(X_2) \to G$ is in fact a group homomorphism with kernel $\{((1,1),(1,1)),((-1,1),(-1,1))\}$.

We show how Theorems 4.1.2 and 4.1.3 follow from Theorem 4.5.4.

PROOF OF THEOREM 4.1.2. By Harish-Chandra's refinement [**32**] of Ado's theorem, $\mathfrak{l}$ has a faithful finite-dimensional representation $\phi : \mathfrak{l} \to \operatorname{End}_K(M)$ such that $\mathfrak{l}$ acts nilpotently on M. Let G be the algebraic group $\mathcal{A}(\phi(\mathfrak{l}))$. By Theorem 4.2.3, $L(G) = \mathfrak{l}$, and G is easily seen to be unipotent. Let H_i be the closed connected subgroup with $L(H_i) = KX_i$. The proof of Lemma 4.5.6 shows that the product map $H_1 \times \ldots \times H_k \to G$ is an isomorphism of varieties. Hence, the conditions of Theorem 4.5.4 are fulfilled, and its conclusion finishes the proof. $\qquad\square$

PROOF OF THEOREM 4.1.3. The proof of Lemma 4.5.6 shows that we can order the Chevalley basis in such a way that the product map from the product of the corresponding one-parameter subgroups into G is an open immersion. In order to apply Theorem 4.5.4, it suffices to check that $\Gamma_{H_\gamma} = \mathbb{Z}$ for all $\gamma \in \Pi$. First, it is contained in $\mathbb{Z}$, as H_γ has only integer eigenvalues on any finite-dimensional $\mathfrak{l}$-module. Conversely, for $n \in \mathbb{Z}$, there exists a cyclic $\mathfrak{l}$-module V on which H_γ has n among its eigenvalues. As G is universal, V is also a G-module, and by Satz II.2.4.1 of [**40**] V is a submodule of $K[G]$. This proves that $\Gamma_{H_\gamma} = \mathbb{Z}$. Application of Theorem 4.5.4 concludes the proof. $\qquad\square$

EXAMPLE 4.5.8. Consider $G = \mathrm{SL}_2$. We identify the Lie algebra $L(G)$ with the vector space spanned by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

endowed with the usual Lie bracket for matrices. Here E and F are nilpotent, and H is semisimple with $\Gamma_H = \mathbb{Z}$. The product map $\mathcal{A}(E) \times \mathcal{A}(H) \times \mathcal{A}(F) \to G$ is an open immersion by the proof of Lemma 4.5.6.

Consider, for U, the affine line $\mathbb{A}^1$ with coordinate x, and the homomorphism $\rho : L(G) \to \operatorname{Der}_K(K[x])$ determined by

$$\rho(E) = -\partial_x, \rho(H) = -2x\partial_x, \text{ and } \rho(F) = x^2\partial_x.$$

This homomorphism satisfies

$$\exp(t\rho(E))(x) = x - t,$$
$$\exp(t\rho(H))(x) = \exp(-2t)x, \text{ and}$$
$$\exp(t\rho(F))(x) = \frac{x}{1 - tx}$$

so that the conditions of Theorem 4.5.4 are fulfilled. It follows that there exists a unique algebraic variety V containing $U = \mathbb{A}^1$ on which G acts morphically, such that the corresponding Lie algebra representation equals ρ. Indeed, this variety V is the projective line $\mathbb{P}^1$, on which G acts by Möbius transformations.

Note that PSL_2, on whose coordinate ring H has $\Gamma_H = 2\mathbb{Z}$, also acts on $\mathbb{P}^1$; this is reflected by the fact that $\exp(t\rho(H))K[x] \subseteq K[x](\exp(2t))$. Note also that the Borel subalgebra $\langle E, H\rangle$ acts locally finitely on $K[x]$, hence the corresponding Borel subgroup of G acts on the affine line by Theorem 4.1.1.

Similarly, the vector fields realizing $\mathfrak{sl}_{n+1}$ in Example 4.5.1 can be used to recover the projective n-space from its affine part, as well as the action of SL_{n+1} on the former.

4.6. Further Research

Although our two main results deal with different cases, Theorem 4.1.1 is more satisfactory than Theorem 4.5.4 in that it constructs the algebraic group from the Lie algebra. This raises the following question: let $\mathfrak{l}$ be a Lie algebra, U an irreducible affine algebraic variety, and $\rho : \mathfrak{l} \to \mathrm{Der}(K[U])$ a Lie algebra homomorphism. Suppose that for all $X \in \mathfrak{l}$, there exist $\lambda_1, \ldots, \lambda_d \in K$ such that

$$\exp(t\rho(X))K[U] \subseteq K(U)(t, \exp(\lambda_1 t), \ldots, \exp(\lambda_d t)).$$

Do there exist an embedding ϕ of $\mathfrak{l}$ into the Lie algebra of an affine algebraic group G, and a morphic action α of G on an algebraic variety V containing U as an open dense subset, such that

$$\rho(X) = -\phi(X)*_\alpha$$

for all $X \in \mathfrak{l}$?

The case where $\mathfrak{l}$ is one-dimensional is already interesting: suppose that $\nabla \in \mathrm{Der}_K(K[U])$ satisfies

$$\exp(t\nabla)K[U] \subseteq K(U)(t, \exp(\lambda_1 t), \ldots, \exp(\lambda_d t)),$$

where the λ_i are independent over $\mathbb{Q}$. Are there mutually commuting derivations $\nabla_0, \nabla_1, \ldots, \nabla_d \in \mathrm{Der}_K(K[U])$ such that $\nabla = \nabla_0 + \nabla_1 + \ldots + \nabla_d$, and $\exp(t\nabla_0)K[U] \subseteq K(U)(t)$ and $\exp(t\nabla_i)K[U] \subseteq K(U)(\exp(\lambda_i t))$ for all i? The answer is yes, and the proof goes along the lines of the proof of Theorem 4.5.4: view $t, \exp(\lambda_1 t_1), \ldots, \exp(\lambda_d t_d)$ as coordinates on $G := G_a \times (G_m)^d$. Then $\exp(-t\nabla)$ is the comorphism of a rational map $\beta : G \times U \to U$, and one can show that the triple (G, U, β) satisfies the conditions of Lemma 4.5.5. Let V and $\alpha : G \times V \to V$ be as in the conclusion of that lemma. Then $\nabla = -(1, \lambda_1, \ldots, \lambda_d)*_\alpha$, and one can take $\nabla_0 = -(1, 0, \ldots, 0)*_\alpha$ and $\nabla_i = -(0, \ldots, 0, \lambda_i, 0, \ldots, 0)*_\alpha$ for $i = 1, \ldots, d$.

As an example, consider the derivation

$$\nabla := (\lambda_1 x + x^2)\partial_x + (\lambda_2 y + xy)\partial_y$$

of $K[x, y]$, where λ_1, λ_2 are independent over $\mathbb{Q}$. It satisfies

$$\exp(t\nabla)x = \frac{\lambda_1 \exp(\lambda_1 t)x}{(1 - \exp(\lambda_1 t))x + \lambda_1} \quad \text{and}$$

$$\exp(t\nabla)y = \frac{\lambda_1 \exp(\lambda_2 t)y}{(1 - \exp(\lambda_1 t))x + \lambda_1}.$$

The right-hand sides are both in $K(x, y, s_1, s_2)$, where $s_i = \exp(\lambda_i t)$ for $i = 1, 2$. If we view the (algebraically independent) s_i as coordinates on G_m^2, the rational map β is given by

$$\beta((s_1^{-1}, s_2^{-1}), (x, y)) = \left(\frac{\lambda_1 s_1 x}{(1 - s_1)x + \lambda_1}, \frac{\lambda_1 s_2 z}{(1 - s_1)x + \lambda_1} \right).$$

Differentiating the group action, we find that

$$-(1,0)*_\alpha = (\frac{1}{\lambda_1}x^2 + x)\partial_x + \frac{1}{\lambda_1}xy\partial_y \text{ and}$$
$$-(0,1)*_\alpha = y\partial_y,$$

so that indeed $\nabla = -\lambda_1(1,0) *_\alpha -\lambda_2(0,1)*_\alpha$.

This one-dimensional case suggests the following line of further research on the case where $\mathfrak{l}$ is higher-dimensional: consider the Lie algebra generated by all vector fields ∇_i as ∇ varies over $\rho(\mathfrak{l})$, and prove that this Lie algebra comes from an algebraic group.

The Adjoint Representation of a Lie Algebra of Vector Fields

5.1. $\mathfrak{D}$ as a $\mathfrak{g}$-Module

A polynomial transitive realization ϕ of a pair $(\mathfrak{g}, \mathfrak{k})$ of codimension n defines a $\mathfrak{g}$-module structure on $\mathfrak{D}^{(n)}$, the Lie algebra of polynomial vector fields in n variables, by $X \cdot Y := [\phi(X), Y]$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{D}^{(n)}$. We will describe the structure of this module in the special case where $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a graded simple Lie algebra over an algebraically closed field K of characteristic 0, $\mathfrak{k} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and ϕ is a graded homomorphism of degree 0. To be precise, denote by $\mathfrak{D}_d$ all derivations of the form $\sum_i f_i \partial_i$ where all f_i are homogeneous polynomials of degree $d + 1$. The $\mathfrak{D}_d$ define a grading on $\mathfrak{D}$, and we require ϕ to map $\mathfrak{g}_d$ into $\mathfrak{D}_d$ for $d = -1, 0, 1$. Such a realization ϕ always exists; indeed, for any basis $\mathbf{Y}$ of $\mathfrak{g}_{-1}$, the Realization Formula of Chapter 2 yields a graded realization $\phi_{\mathbf{Y}}$. Moreover, from the proof of Theorem 2.3.4 it is clear that a graded realization is unique up to linear coordinate changes; it is described in a coordinate-free manner on page 74.

To appreciate the results of this chapter, consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ is graded by $\mathfrak{g}_{-1} = KE$, $\mathfrak{g}_0 = KH$, and $\mathfrak{g}_1 = KF$, where E, H, F is the Chevalley basis of $\mathfrak{sl}_2$. The embedding ϕ from $\mathfrak{g}$ into $\mathfrak{D}^{(1)}$ is given by

$$E \mapsto \partial_x, \; H \mapsto -2x\partial_x, \text{ and } F \mapsto -x^2\partial_x;$$

see also Example 2.3.2. Of course, $\mathfrak{D}$ has $\phi(\mathfrak{g})$ as a submodule, and it is readily seen that the quotient $\mathfrak{D}/\phi(\mathfrak{g})$ is an irreducible module generated by the highest weight vector $x^3\partial_x + \phi(\mathfrak{g})$ of weight -4.

On the other hand, consider the case where $\mathfrak{g} = \mathfrak{sl}_4$, with grading defined by the block decomposition

$$\left(\begin{array}{c|c} \mathfrak{g}_0 & \mathfrak{g}_{-1} \\ \hline \mathfrak{g}_1 & \mathfrak{g}_0 \end{array} \right),$$

where each block has size 2×2. Hence, $\mathfrak{g}_{-1}$ is the subalgebra of $\mathfrak{g}$ consisting of those matrices in which all blocks other than the upper right one are zero, etc. Clearly, $\mathfrak{g}_{-1}$ has dimension 4; a choice of basis $\partial_1, \partial_2, \partial_3, \partial_4$ of $\mathfrak{g}_{-1}$, with dual basis x_1, x_2, x_3, x_4, defines an embedding $\phi : \mathfrak{g} \to \mathrm{Der}_K(K[x_1, x_2, x_3, x_4]) = \mathfrak{D}$ of graded Lie algebras. By computer calculations using the function `Blattner` of Appendix B to compute ϕ, we find that the dimension of $[\phi(\mathfrak{g}_1), \mathfrak{D}_4] \subseteq \mathfrak{D}_5$ equals 332, instead of $\dim \mathfrak{D}_5 = 4\binom{6+4-1}{4-1} = 336$. By the PBW-theorem we have $U(\mathfrak{g}) = U(\mathfrak{g}_1)U(\mathfrak{g}_0)U(\mathfrak{g}_{-1})$, so the homogeneous part of

$U(\mathfrak{g})\mathfrak{D}_4$ of degree 5 equals

$$(U(\mathfrak{g}_1)(\bigoplus_{d=-1}^{4} \mathfrak{D}_d))_5 = [\phi(g)_1, \mathfrak{D}_4],$$

and is therefore strictly contained in $\mathfrak{D}_5$; hence, $\mathfrak{D}_4$ generates a $\mathfrak{g}$-submodule properly contained in $\mathfrak{D}$ which projects onto a proper submodule of $\mathfrak{D}/\phi(\mathfrak{g})$.

The two examples above are representative for the main results of this chapter, which are Theorem 5.4.2, presenting a formula for the multiplicities of the irreducible quotients in a composition chain of $\mathfrak{D}$ as a $\mathfrak{g}$-module; and Conjectures 5.4.4 and 5.4.5, which describe in detail which irreducible quotients do really occur.

Our method is best illustrated by the example with $\mathfrak{g} = \mathfrak{sl}_2$. Consider the Verma module $M = U(\mathfrak{sl}_2) \otimes_{\langle E,H \rangle_K} Kv_3$ of highest weight $3-1 = 2$; see page 76 for a definition and an explanation why the highest weight vector is labelled with 3 rather than 2. The element $F^3 \otimes v_3 \in M$ generates an irreducible submodule N of M of highest weight -4, and M/N is the irreducible module of highest weight 2. So, it seems that M is in some sense dual to $\mathfrak{D}$. Indeed, we will show that, in the simple-graded case, $\mathfrak{D}$ is always dual to a generalized Verma module M, which implies that $\mathfrak{D}$ and M have the same irreducible quotients in their respective composition chains. For M we derive a Kostant-type formula that can be used to determine those quotients, as well as their multiplicities. Experiments with this formula in the computer algebra program LiE [**57**] yield empirical evidence for a conjecture describing them in detail; we prove part of this conjecture.

5.2. Preliminaries

Simple Graded Lie Algebras of Depth One. Let $\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{g}_d$ be a graded Lie algebra over K with the property $\mathfrak{g}_{-d} = 0$ for d sufficiently large. The smallest $d \geq 0$ with $\mathfrak{g}_{-e} = 0$ for all $e > d$ is called the *depth* of $\mathfrak{g}$; assume that it is 1. Define $V := \mathfrak{g}_{-1}$ and set

$$\mathfrak{D} := \bigoplus_{d=-1}^{\infty} \mathfrak{D}_d, \text{ where } \mathfrak{D}_d := S^{d+1}(V^*) \otimes V.$$

Here $S^e(V^*)$ denotes the e-th symmetric power of V^*; the direct sum of these spaces is the symmetric algebra $S(V^*)$. Let $(v, f) \mapsto v(f)$ denote the natural map from $V \times S^{d+1}(V^*)$ to $S^d(V^*)$, and, similarly, from $V \times S(V^*)$ to $S(V^*)$. Then $\mathfrak{D}$ is a Lie algebra with respect to the Lie bracket determined by

$$[f_1 \otimes v_1, f_2 \otimes v_2] := f_1 v_1(f_2) \otimes v_2 - f_2 v_2(f_1) \otimes v_1.$$

For $X \in \mathfrak{g}_d$, define $\phi(X) : V^{d+1} \to V$ by

$$\phi(X)(v_1, v_2, \ldots, v_{d+1}) := \frac{1}{(d+1)!} \operatorname{ad}(v_1) \operatorname{ad}(v_2) \cdots \operatorname{ad}(v_{d+1}) X,$$

for $v_1, \ldots, v_{d+1} \in \mathfrak{g}_{-1}$. As $\mathfrak{g}_{-1}$ is Abelian, $\phi(X)$ defines an element of $S^{d+1}(V^*) \otimes V$. Now ϕ, thus defined on each $\mathfrak{g}_d$, can be shown to extend linearly to a homomorphism of Lie algebras. Indeed, with respect to a basis $\mathbf{Y} = (Y_1, \ldots, Y_n)$ of $\mathfrak{g}_{-1}$, ϕ is easily seen to be described by the Realization Formula of Chapter 2.

In [**45**], Morozov describes all simple graded Lie algebras of depth 1. First, one checks that the Killing form pairs $\mathfrak{g}_{-d}$ and $\mathfrak{g}_d$ non-degenerately, for all d. This implies that $\mathfrak{g}_d = 0$ for $d > 1$, and that $\mathfrak{g}_1$ is dual to $\mathfrak{g}_{-1}$ as a $\mathfrak{g}_0$-module.

THEOREM 5.2.1 (Morozov). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple graded Lie algebra over K. Then one can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a fundamental system Π in the root system $\Delta \subseteq \mathfrak{h}^*$, and a root $\beta \in \Pi$ such that for $i = -1, 0, 1$,*

$$(14) \qquad \mathfrak{g}_i = \bigoplus_{\substack{\alpha = \sum_{\gamma \in \Pi} n_\gamma \gamma \in \Delta, \\ n_\beta = -i}} \mathfrak{g}_\alpha.$$

Here we write $\mathfrak{g}_\alpha$ for the $\mathfrak{h}$-weight space in $\mathfrak{g}$ of weight α; see below for a definition. Note that in particular $\mathfrak{h} \subseteq \mathfrak{g}_0$ and $\mathfrak{g}_{\pm\beta} \subseteq \mathfrak{g}_{\mp 1}$. The minus sign in (14) may appear somewhat unnatural, but is chosen so that positive root vectors of $\mathfrak{g}$ act nilpotently on $\mathfrak{D}$ in the realization described above. Note also that $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} = \mathfrak{p}_\beta$ in the notation of page 7.

If $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a graded simple Lie algebra, we shall say that $\mathfrak{g}$ is of type (X_n, i) if the Cartan type of $\mathfrak{g}$ is X_n and β corresponds to the i-th node in the Dynkin diagram; here we use the standard labelling of [**7**]. The list of pairs (X_n, i) for which the i-th simple root has coefficient 1 in the highest root is well known: (A_n, i) for $i = 1, \ldots, n$, $(B_n, 1)$, (C_n, n), $(D_n, 1)$, $(D_n, n-1)$, (D_n, n), $(E_6, 1)$, $(E_6, 6)$, and $(E_7, 7)$; for instance, the second example of Section 5.1 is of type $(A_3, 2)$. The fundamental weight corresponding to such i is called a *cominuscule weight*, referring to the fact that the corresponding weight of the dual Lie algebra is minuscule ([**8**], Chapitre 8).

Duality for Weight Modules. In the remainder of this chapter $\mathfrak{g}$ is a simple Lie algebra over K, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\Delta \subseteq \mathfrak{h}^*$ is the root system with respect to $\mathfrak{h}$, Π is a fundamental system in Δ, and $\Delta_\pm$ are the corresponding sets of positive and negative roots, respectively. The Borel subalgebras $\mathfrak{b}_\pm$ corresponding to Π, and their nilpotent radicals $\mathfrak{n}_\pm$, are given by

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha \text{ and } \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm;$$

we also write $\mathfrak{b}$ for $\mathfrak{b}_+$. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ denote the *Chevalley involution* corresponding to Π, i.e., the unique automorphism of $\mathfrak{g}$ extending $X_\gamma \mapsto X_{-\gamma}$ $(\gamma \in \pm\Pi)$ and $H_\gamma \mapsto -H_\gamma$ $(\gamma \in \Pi)$, where the $X_{\pm\gamma}$ and H_γ $(\gamma \in \Pi)$ are Chevalley generators of $\mathfrak{g}$.

An $\mathfrak{h}$-weight module is an $\mathfrak{h}$-module M with the following properties: the space

$$M_\lambda := \{v \in M \mid hv = \langle h, \lambda \rangle v \text{ for all } h \in \mathfrak{h}\}$$

of weight λ is finite-dimensional for all $\lambda \in \mathfrak{h}^*$, and $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$. For such M, we write $\mathrm{ch}(M)$ for the function $\mathfrak{h}^* \to \mathbb{N}$ that sends λ to $\dim M_\lambda$. This function is called the *character* of M. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then an $\mathfrak{h}$-weight $\mathfrak{k}$-module is a $\mathfrak{k}$-module that is also an $\mathfrak{h}$-weight module.

Let $\mathcal{C}$ denote the category of all $\mathfrak{h}$-weight $\mathfrak{g}$-modules, with $\mathfrak{g}$-module homomorphisms as morphisms. If M is an object in $\mathcal{C}$, and f is an element of the vector space $M^* = \mathrm{Hom}_K(M, K)$ dual to M, then we define the *support* of f by

$$\mathrm{supp}(f) := \{\lambda \in \mathfrak{h}^* \mid f|_{M_\lambda} \neq 0\}.$$

Now the space

$$M^\vee := \{f \in M^* \mid \operatorname{supp}(f) \text{ is finite}\}$$

is a $\mathfrak{g}$-module with action defined by

$$\langle m, X \cdot f \rangle = -\langle \sigma(X)m, f \rangle, \ m \in M, f \in M^\vee,$$

and it is easily verified that $M^\vee$ is again an object in $\mathcal{C}$. The σ-twist is convenient when studying the full subcategory $\mathcal{O}$ of $\mathcal{C}$ consisting of finitely generated modules on which $\mathfrak{n}_+$ acts locally finitely (see page 57 for a definition). Standard references for properties of this category are [**15**] and [**36**]; to mention a few: $\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums, and all objects in $\mathcal{O}$ have a finite composition chain. Moreover, if we denote by Q_+ the set of linear combinations of Π with non-negative integer coefficients, and if M is an object in $\mathcal{O}$, then $\operatorname{supp}(\operatorname{ch}(M))$ is contained in the union of a finite number of sets of the form $\lambda - Q_+$ with $\lambda \in \mathfrak{h}^*$.

PROPOSITION 5.2.2. *Let M be an object in $\mathcal{C}$.*
(1) *The modules $(M^\vee)^\vee$ and M are isomorphic.*
(2) *The characters of M and $M^\vee$ coincide.*
(3) *The map sending a submodule of M to its annihilator in $M^\vee$ is a bijection.*
(4) *If M has a finite composition chain, then so has $M^\vee$.*
(5) *If M is in $\mathcal{O}$, then so is $M^\vee$, and they have the same irreducible factors with the same multiplicities.*

PROOF. Verification of statements (1)–(4) is straightforward. To prove (5), let M be an object of $\mathcal{O}$. By (4), the dual $M^\vee$ has a finite composition chain; in particular, it is finitely generated. By (2), the support of the character of $M^\vee$ is contained in the union of a finite number of sets of the form $\lambda - Q_+$. This implies that $\mathfrak{n}_+$, having $\mathfrak{h}$-roots that are non-zero elements of Q_+, acts locally nilpotently on $M^\vee$. Hence, $M^\vee$ is an object in $\mathcal{O}$ as claimed. The last statement follows from (2) and the fact that the map $M \mapsto \operatorname{ch}(M)$ induces a monomorphism from the Grothendieck group of $\mathcal{O}$ ([**36**], Section 1.11). $\square$

Verma Modules. The category $\mathcal{O}$ contains the *Verma modules*, which are defined as follows. Set

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha,$$

let $\lambda \in \mathfrak{h}^*$, and let Kv_λ be the $\mathfrak{b}$-module defined by

$$hv_\lambda = \langle h, \lambda - \rho \rangle v_\lambda \text{ for } h \in \mathfrak{h}, \text{ and } \mathfrak{n}_+ v_\lambda = 0.$$

Now the induced module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} Kv_\lambda$$

is the Verma module of highest weight $\lambda - \rho$. We recall a few properties of Verma modules from the standard references [**15**] and [**36**]: first, $M(\lambda)$ is universal among the $\mathfrak{g}$-modules generated by a highest weight vector of weight $\lambda - \rho$, and it has a unique irreducible quotient, denoted by $L(\lambda)$. All irreducible modules in $\mathcal{O}$ are of the form $L(\lambda)$ for some λ. Moreover, $\dim \operatorname{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda)) \leq 1$ for all $\lambda, \mu \in \mathfrak{h}^*$, and any non-trivial homomorphism from $M(\mu)$ to $M(\lambda)$ is injective. We may therefore view $M(\mu)$ as a submodule of $M(\lambda)$ whenever $\dim \operatorname{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda)) = 1$.

The parameterization of Verma modules by their highest weight plus ρ seems somewhat unnatural—indeed, if $\lambda = \mathbf{m} \in \mathbb{N}^n$ with respect to the basis of fundamental weights, then $L(\lambda) \cong V_{\mathbf{m}-(1,\dots,1)}$ in the notation of Section 3.3—but it is justified by the BGG-criterion, which is a necessary and sufficient condition on λ and μ for $M(\mu)$ to be a submodule of $M(\lambda)$. We shall state the criterion only for the case that we shall need, namely the case where λ is (strictly) dominant and integral.

To this end, let $(\cdot, \cdot)$ be the symmetric bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta$, we define $\alpha^\vee := \frac{2}{(\alpha, \alpha)} \alpha$, and a reflection s_α on $\mathfrak{h}^*$ by

$$s_\alpha \lambda = \lambda - (\lambda, \alpha^\vee) \alpha.$$

Let P^{++} be the set of all $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \gamma^\vee)$ is a positive integer for all $\gamma \in \Pi$. Taking into account that $L(\lambda)$ has highest weight $\lambda - \rho$, we have that P^{++} is precisely the set of λ for which $L(\lambda)$ is finite-dimensional [35]. Next, let W be the Weyl group of Δ, i.e., the group generated by the s_α for $\alpha \in \Delta$, considered as a Coxeter group with generators s_γ for $\gamma \in \Pi$. Let $l : W \to \mathbb{N}$ be the length function of W, and denote by $\leq$ the Bruhat order on W, i.e., the reflexive and transitive closure of the union of the relations $\to_\alpha$ $(\alpha \in \Delta)$ defined by $w_1 \to_\alpha w_2 \Leftrightarrow (s_\alpha w_1 = w_2) \wedge (l(w_1) < l(w_2))$; see [14] and [59] for properties of this order.

THEOREM 5.2.3 (BGG-criterion). *Let $\lambda \in P^{++}, \mu \in \mathfrak{h}^*$, and $w_1 \in W$. Then the following are equivalent:*

(1) *$L(\mu)$ occurs as a quotient in a composition chain of $M(w_1(\lambda))$,*
(2) *$M(\mu)$ is a submodule of $M(w_1(\lambda))$, and*
(3) *$\mu = w_2(\lambda)$ for some $w_2 \in W$ with $w_1 \leq w_2$.*

The implication $3 \Rightarrow 2$ is due to Verma ([58]; see also [3] for a proof without case-by-case analysis). The implication $2 \Rightarrow 1$ is clear, and the implication $1 \Rightarrow 3$ is the main contribution of Bernstein, Gel'fand, and Gel'fand in [3], which explains the name of the above theorem.

Verma conjectured that all submodules of a Verma module $M(\lambda)$ are spanned by the Verma modules contained in it. Bernstein, Gel'fand and Gel'fand, however, gave a counterexample to this conjecture in [3]. This example shows that some irreducible quotients have multiplicities larger than 1 in a composition chain of $M(\lambda)$. For a long time, it was unclear what those multiplicities were, but in [39] Kazhdan and Lusztig defined certain polynomials P_{w_1, w_2} for $w_1, w_2 \in W$, and conjectured that

$$(M(w_1(\lambda)) : L(w_2(\lambda))) = P_{w_1, w_2}(1),$$

where $(M : L)$ denotes the multiplicity of L among the irreducible quotients in a composition chain of M. This conjecture was proved independently by Beilinson and Bernstein in [2] and by Brylinski and Kashiwara in [12].

5.3. Generalized Verma Modules and a Kostant-type Formula

Verma modules are induced from irreducible $\mathfrak{b}$-modules. More generally, we can start with a finite-dimensional irreducible module V for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, and induce it to a module $M_{\mathfrak{p}}(V)$ of $\mathfrak{g}$. In taking $\mathfrak{p} = \mathfrak{b}$, we retrieve the ordinary Verma modules.

More specifically, let Π_0 be a subset of Π, let $\Delta_0 \subseteq \Delta$ be the set of all roots that are linear combinations of Π_0, and let $\Delta_{\pm 1} := \Delta_\pm \setminus \Delta_0$. Define

$$\mathfrak{n}_{0\pm} := \bigoplus_{\alpha \in \Delta_0 \cap \Delta_\pm} \mathfrak{g}_\alpha, \ \ \mathfrak{g}_0 := \mathfrak{n}_{0-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0+}, \ \ \mathfrak{u}_\pm := \bigoplus_{\alpha \in \Delta_{\pm 1}} \mathfrak{g}_\alpha, \ \text{and} \ \mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{u}_+.$$

In the notation of page 7 we have $\mathfrak{p} = \mathfrak{p}_{\Pi_0}$. The derived subalgebra $\mathfrak{g}'_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple with Cartan subalgebra $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}'_0$ and root system $\Delta_0|_{\mathfrak{h}_0}$ (unless $\Pi_0 = \emptyset$). The space

$$\mathfrak{h}_c := \{h \in \mathfrak{h} \mid \langle h, \Pi_0 \rangle = 0\}$$

is a vector space complement of $\mathfrak{h}_0$ in $\mathfrak{h}$, and it equals the center of $\mathfrak{g}_0$. Finally, denote by $W_0 \subseteq W$ the group generated by the s_γ with $\gamma \in \Pi_0$; it is the Weyl group of Δ_0.

Now let V be a $\mathfrak{p}$-module, and consider the $\mathfrak{g}$-module

$$M_{\mathfrak{p}}(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

This module is generated by a $\mathfrak{p}$-submodule isomorphic to V, and by the PBW-theorem it is a free $U(\mathfrak{u}_-)$-module. Moreover, these two properties characterize $M_{\mathfrak{p}}(V)$.

In particular, if V is an irreducible finite-dimensional $\mathfrak{g}_0$-module, then we can turn V into a $\mathfrak{p}$-module by setting $\mathfrak{u}_+ V = 0$. By Schur's lemma, $\mathfrak{h}_c$ acts on V by a scalar $\mu \in \mathfrak{h}_c^*$. In fact, $M_{\mathfrak{p}}(V)$ splits as a direct sum of finite-dimensional $\mathfrak{h}_c$-weight spaces, each of which is a $\mathfrak{g}_0$-submodule; in particular, $(M_{\mathfrak{p}}(V))_\mu \cong V$. Moreover, if M is any other $\mathfrak{g}$-module whose highest $\mathfrak{h}_c$-weight is μ, with a weight space M_μ that is isormophic to V as a $\mathfrak{g}_0$-module and that generates the $\mathfrak{g}$-module M, then M is a quotient of $M_{\mathfrak{p}}(V)$.

The modules $M_{\mathfrak{p}}(V)$ are called *generalized Verma modules* [**44**], [**54**]. We shall need the following analogue of Kostant's theorem for finite-dimensional irreducible modules ([**35**], Section VIII.5).

PROPOSITION 5.3.1. *Let V be an irreducible finite-dimensional $\mathfrak{g}_0$-module, and let $\lambda - \rho \in \mathfrak{h}^*$ be its highest weight. Then we have the identity*

$$(15) \qquad\qquad [M_{\mathfrak{p}}(V)] = \sum_{w_0 \in W_0} (-1)^{l(w_0)} [M(w_0(\lambda))]$$

in the Grothendieck group of $\mathcal{O}$.

We need some notation for the proof. Following [**3**], we denote by $\mathcal{E}$ the set of all functions $\mathfrak{h}^* \to \mathbb{Z}$ whose support is contained in a finite union of sets of the form $\lambda - Q_+$. Then $\mathcal{E}$ becomes a commutative $\mathbb{Z}$-algebra when endowed with the convolution

$$u_1 * u_2(\lambda) := \sum_{\mu \in \mathfrak{h}^*} u_1(\mu) u_2(\lambda - \mu).$$

For $\lambda \in \mathfrak{h}^*$ the element $e^\lambda \in \mathcal{E}$ is defined by $e^\lambda(\mu) := \delta_{\lambda,\mu}$. Note that $e^\lambda * e^\mu = e^{\lambda+\mu}$, and if M is an $\mathfrak{h}$-weight module, then

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathfrak{h}^*} \dim(M_\lambda) e^\lambda.$$

If this is an element of $\mathcal{E}$, and if N is another $\mathfrak{h}$-weight module with $\mathrm{ch}(N) \in \mathcal{E}$, then $M \otimes N$ is also an $\mathfrak{h}$-weight module, and

$$\mathrm{ch}(M \otimes N) = \mathrm{ch}(M) * \mathrm{ch}(N).$$

PROOF OF PROPOSITION 5.3.1. From the fact that $M_{\mathfrak{p}}(V) \cong U(\mathfrak{u}_-) \otimes V$ as $\mathfrak{h}$-modules, we find

$$\mathrm{ch}(M_{\mathfrak{p}}(V)) = \mathrm{ch}(U(\mathfrak{u}_-)) * \mathrm{ch}(V).$$

By Kostant's formula for the weight multiplicities in the irreducible $\mathfrak{g}_0$-module V we have

$$\mathrm{ch}(V) = \sum_{w_0 \in W_0} (-1)^{l(w_0)} e^{w_0(\lambda)-\rho} * \mathrm{ch}(U(\mathfrak{n}_{0-})).$$

Combining these formulas, we find

$$\begin{aligned}
\mathrm{ch}(M_{\mathfrak{p}}(V)) &= \sum_{w_0 \in W_0} (-1)^{l(w_0)} e^{w_0(\lambda)-\rho} * \mathrm{ch}(U(\mathfrak{n}_{0-})) * \mathrm{ch}(U(\mathfrak{u}_-)) \\
&= \sum_{w_0 \in W_0} (-1)^{l(w_0)} e^{w_0(\lambda)-\rho} * \mathrm{ch}(U(\mathfrak{n}_-)) \\
&= \sum_{w_0 \in W_0} (-1)^{l(w_0)} \mathrm{ch}(M(w_0(\lambda))),
\end{aligned}$$

where we used the fact that $U(\mathfrak{n}_{0-}) \otimes U(\mathfrak{u}_-) \cong U(\mathfrak{n}_-)$ as $\mathfrak{h}$-modules. Now the lemma follows from the fact that the map sending $[M]$ to $\mathrm{ch}(M)$ extends to a monomorphism of the Grothendieck group of $\mathcal{O}$ into $\mathcal{E}$, ([**36**], Section 1.11). $\square$

The following proposition gives an alternative description of generalized Verma modules induced from finite-dimensional irreducible $\mathfrak{g}_0$-modules.

PROPOSITION 5.3.2. *Let V be a finite-dimensional irreducible $\mathfrak{g}_0$-module, and let $\lambda - \rho \in \mathfrak{h}^*$ be its highest weight. Then*

$$M_{\mathfrak{p}}(V) \cong M(\lambda)/\sum_{\gamma \in \Pi_0} M(s_\gamma(\lambda)).$$

PROOF. First, $M_{\mathfrak{p}}(V)$ is generated by $1 \otimes \tilde{v}_\lambda$, where $\tilde{v}_\lambda$ is the highest weight vector of the $\mathfrak{g}_0$-module V. The vector $1 \otimes \tilde{v}_\lambda$ is a highest weight vector for $\mathfrak{g}$ of weight $\lambda - \rho$, so $M_{\mathfrak{p}}(V)$ is a quotient of $M(\lambda)$ by the universal property of Verma modules, i.e., there exists a surjective morphism $\pi : M(\lambda) \to M_{\mathfrak{p}}(V)$ of $\mathfrak{g}$-modules.

Let $N_1 \subseteq M(\lambda)$ be the kernel of π, and let $1 \otimes v_\lambda$ be the highest weight vector of $M(\lambda)$. The $\mathfrak{g}_0'$-submodule N_2 of $M(\lambda)$ generated by $1 \otimes v_\lambda$ is isomorphic to the Verma module $M(\lambda|_{\mathfrak{h}_0})$ for $\mathfrak{g}_0'$, and π maps N_2 onto $1 \otimes V$. Hence, N_1 contains the maximal $\mathfrak{g}_0'$-submodule N_3 of N_2. This module is generated by the elements $e_\gamma^{(\lambda,\gamma^\vee)} 1 \otimes v_\lambda$ for $\gamma \in \Pi_0$ ([**34**], Theorem 21.4); these generate $\mathfrak{g}$-submodules isomorphic to $M(s_\gamma(\lambda))$. This shows that N_1 contains the $\mathfrak{g}$-module

$$N_4 := \sum_{\gamma \in \Pi_0} M(s_\gamma(\lambda)).$$

Hence, $M_{\mathfrak{p}}(V)$ is a quotient of $M(\lambda)/N_4$. Conversely, the $\mathfrak{g}_0$-submodule N_5 of $M(\lambda)/N_4$ corresponding to $\mathfrak{h}_c$-weight $(\lambda - \rho)|_{\mathfrak{h}_c}$ generates $M(\lambda)/N_4$. Moreover, N_5 is isomorphic to V and the e_α with $\alpha \in \Delta_1$ annihilate it. By the universal property of $M_{\mathfrak{p}}(V)$ we find that $M(\lambda)/N_4$ is a quotient of $M_{\mathfrak{p}}(V)$. We have thus found a sequence

$$M_{\mathfrak{p}}(V) \to M(\lambda)/N_4 \to M_{\mathfrak{p}}(V)$$

of epimorphisms. Their concatenation is an epimorphism of $\mathfrak{g}$-modules, hence an iso-morphism as $M_{\mathfrak{p}}(V)$ admits only scalar homomorphisms [**44**]. The proposition now follows.   □

This proposition suggests an alternative proof of Proposition 5.3.1. Indeed, let V and λ be as in that proposition, and *suppose* that the following holds.

(∗)   The set $P_0 := \{\sum_{w \in T} M(w(\lambda)) \mid T \subseteq W_0\}$ is closed under taking intersections.

We show that this implies Proposition 5.3.1. To this end, consider the map sending $M \in P_0$ to $\{w \in W_0 \mid M(w(\lambda)) \subseteq M\}$. By the BGG-criterion this is a bijection from P_0 to the set

$$P_0' := \{T \subseteq W_0 \mid \forall t \in T, w \in W_0 : t \leq w \Rightarrow w \in T\},$$

and under condition (∗) it even is an isomorphism $(P_0, +, \cap) \to (P_0', \cup, \cap)$ of lattices. The latter is finite and distributive, hence so is the former. The proof of the following lemma on such lattices is straightforward.

LEMMA 5.3.3. *Let $(Q, \sqsubseteq)$ be a finite distributive lattice, and define the relation $\sqsubset$ on Q by*

$$p \sqsubset q :\Leftrightarrow p \sqsubseteq q \text{ and } p \neq q.$$

Let F be the free Abelian group generated by Q. Define $A := F/N$, where N is the subgroup generated by the minimum $\bot$ of Q, and the elements of the form $p \sqcup q - p - q + p \sqcap q$. Denote the image of $q \in Q$ in A by $[q]$. Then the following identity holds for all $p \in Q$:

$$[p] = \sum_{q \sqsubseteq p} ([q] - [\sup\{r \mid r \sqsubset q\}]).$$

Note that, in the sum of the lemma, only those q for which $\sup\{r \mid r \sqsubset q\} \neq q$ yield a non-zero contribution. Now apply this lemma to P_0 to find the identity

$$[M(w(\lambda))] = \sum_{w_1 \geq w} \left([M(w_1(\lambda))] - \left[\sum_{w_2 > w_1} M(w_2(\lambda)) \right] \right)$$

in the Grothendieck group of $\mathcal{O}$; here w and all indices are taken in W_0. Indeed, the other terms that would appear in the sum on the right-hand side according to the lemma, are zero by virtue of the remark above. By Verma's Möbius inversion on W_0 [**59**] we have

$$[M(w(\lambda))] - \left[\sum_{w_0 > w} M(w_0(\lambda)) \right] = \sum_{w_0 \geq w} (-1)^{l(w)+l(w_0)} [M(w_0(\lambda))].$$

In particular, taking $w = e$ and using Proposition 5.3.2, we retrieve (15).

We conclude that condition (∗) yields a transparent proof of a Kostant-type for-mula for generalized Verma modules, implying Kostant's formula for finite-dimensional irreducible modules. In fact, Verma's motivation for studying Möbius inversion over Coxeter groups in [**59**] was precisely this, but it is not clear from the literature whether he succeeded.

Validity of $(*)$ for any Π_0 would follow from its validity for $\Pi_0 = \Pi$. Indeed, suppose that it holds for Π, let Π_0 be any subset of Π, and let T_1, T_2 be subsets of the corresponding Weyl subgroup W_0. By $(*)$ for Π, the $\mathfrak{g}$-module

$$\left(\sum_{w \in T_1} M(w(\lambda)) \right) \cap \left(\sum_{w \in T_2} M(w(\lambda)) \right)$$

is spanned by the Verma modules that it contains. By the BGG-criterion, these are the submodules $M(w(\lambda))$ of $M(\lambda)$ for which $w \in W$ has the property that there exist $w_i \in T_i$, $(i = 1, 2)$ such that $w_1, w_2 \leq w$. It remains to show that for such w, w_1, w_2 there exists a $w_0 \in W_0$ for which $w_1, w_2 \leq w_0 \leq w$, so that $M(w(\lambda)) \subseteq M(w_0(\lambda))$. The following lemma will be shown to imply this.

LEMMA 5.3.4. *Let W be a Coxeter group with finite set of distinguished generators S. Let $\mathbf{r} = (r_1, \ldots, r_q)$ be any sequence of elements of S, and define*

$$U(\mathbf{r}) := \{ r_{i_1} \cdots r_{i_k} \mid k \in \mathbb{N}, 1 \leq i_1 < i_2 < \ldots < i_k \leq q \}.$$

Then $U(\mathbf{r})$ has a unique maximal element with respect to the Bruhat order.

Scott Murray helped me out with the following proof.

PROOF. Proceed by induction on q. For $q = 0$ the lemma holds trivially; now suppose that it holds for certain $q \geq 0$, and let $\mathbf{r} = (r_1, \ldots, r_{q+1})$ be a sequence of elements in S. By the induction hypothesis, $U((r_1, \ldots, r_q))$ has a unique maximal element w. Let w' be the longest of the elements w and wr_{q+1}; we claim that w' is the maximum of $U(\mathbf{r})$.

Indeed, denoting the length function on W by l, suppose that $l(wr_{q+1}) < l(w)$ so that $w' = w$, and let $u \in U(\mathbf{r})$. Then u has a reduced expression $(r_{i_1}, \ldots, r_{i_k})$ for some k and $1 \leq i_1 < \ldots < i_k \leq q+1$ ([**7**], Chapitre IV, §1, no. 4, Lemme 2). If $i_k \neq q+1$, then $u \in U((r_1, \ldots, r_q))$ and $u \leq w$ by definition of w. If $i_k = q+1$, then $l(ur_{q+1}) < l(u)$; together with $l(wr_{q+1}) < l(w)$ this implies ([**59**], Lemma) that $u \leq w$ is equivalent to $ur_{q+1} \leq w$. The latter inequality holds by definition of w, hence so does the former.

The case where $l(wr_{q+1}) < l(w)$ can be treated similarly, and this concludes the proof of the lemma. $\square$

In our application, let $\mathbf{r} = (r_1, \ldots, r_q)$ be the sequence of fundamental reflections in W_0 obtained by deleting the fundamental reflections in $W \setminus W_0$ from a reduced expression for w. Then $w_1, w_2 \in U(\mathbf{r})$ [**14**], and we may take w_0 to be the maximal element of $U(\mathbf{r})$. This concludes the proof of our claim that validity of $(*)$ for $\Pi_0 = \Pi$ implies validity of $(*)$ for all $\Pi_0 \subseteq \Pi$.

It seems unknown, even to experts, whether $(*)$ holds for $\Pi_0 = \Pi$. In an e-mail Anthony Joseph let me know that proving $(*)$ seemed very hard to him, so that it would not be worth the trouble for merely reproving Kostant's formula. However, the fact that $(*)$ implies that formula is reassuring, and in my opinion a good motivation for further research into this question.

5.4. The Composition Factors of $\mathfrak{D}$

To prove our main result we characterize $\mathfrak{D}$ as follows.

PROPOSITION 5.4.1. *Let $\mathfrak{g}$ be as in Theorem 5.2.1, and set $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$. Then the $\mathfrak{g}$-module $\mathfrak{D}$ of page 74 is isomorphic to $M_{\mathfrak{p}}(\mathfrak{g}_{-1})^\vee$.*

PROOF. Let $\mathfrak{h}, \Pi, \Delta$, and β be as in the conclusion of Theorem 5.2.1, and set

$$\Delta_{\pm 1} := \left\{ \sum_{\gamma \in \Pi} n_\gamma \gamma \in \Delta \mid n_\beta = \pm 1 \right\}.$$

For $\alpha \in \Delta_1$ let $\partial_\alpha \in \mathfrak{g}$ be a root vector of weight α. Together, these form a basis of $\mathfrak{g}_{-1}$; let $\mathbf{X} = (X_\alpha)_{\alpha \in \Delta_1}$ be the corresponding dual basis of $\mathfrak{g}_1$. Then we have $\sigma(X_\alpha) = c_\alpha \partial_\alpha$ for some non-zero $c_\alpha \in K$. On the other hand, we may view the X_α as elements of $\mathfrak{g}_{-1}^*$, and hence as generators of $S(\mathfrak{g}_{-1}^*)$. Now $(\mathbf{X}^{\mathbf{m}} \partial_\beta)_{\mathbf{m} \in \mathbb{N}^{\Delta_1}, \beta \in \Delta_1}$ is a basis of $\mathfrak{D}$; let $(f_{\mathbf{m},\beta})$ denote the dual basis of $\mathfrak{D}^\vee$. For $\alpha, \beta, \gamma \in \Delta_1$ and $\mathbf{m}, \mathbf{r} \in \mathbb{N}^{\Delta_1}$ compute

$$\begin{aligned}
\langle \mathbf{X}^{\mathbf{r}} \partial_\gamma, X_\alpha \cdot f_{\mathbf{m},\beta} \rangle &= -\langle \sigma(X_\alpha) \mathbf{X}^{\mathbf{r}} \partial_\gamma, f_{\mathbf{m},\beta} \rangle \\
&= -c_\alpha \langle [\partial_\alpha, \mathbf{X}^{\mathbf{r}} \partial_\gamma], f_{\mathbf{m},\beta} \rangle \\
&= -c_\alpha \langle r_\alpha \mathbf{X}^{\mathbf{r} - \mathbf{e}_\alpha} \partial_\gamma, f_{\mathbf{m},\beta} \rangle \\
&= -(m_\alpha + 1) c_\alpha \delta_{\mathbf{m} + \mathbf{e}_\alpha, \mathbf{r}} \delta_{\beta, \gamma},
\end{aligned}$$

where $\mathbf{e}_\alpha$ denotes the standard basis vector of $\mathbb{N}^{\Delta_1}$ corresponding to $\alpha \in \Delta_1$. Hence, $X_\alpha \cdot f_{\mathbf{m},\beta} = -(m_\alpha + 1) c_\alpha f_{\mathbf{m} + \mathbf{e}_\alpha, \beta}$. This shows that $\mathfrak{g}_1$ acts freely on $\mathfrak{D}^\vee$, and that the latter module is generated by its $\mathfrak{g}_0$-submodule spanned by $(f_{0,\beta})_{\beta \in \Delta_1}$. This module is dual to the Chevalley twist of the finite-dimensional $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$, whence isomorphic to $\mathfrak{g}_{-1}$. We have thus proved that $\mathfrak{D}^\vee$ is isomorphic to $M_{\mathfrak{p}}(\mathfrak{g}_{-1})$; the proposition now follows from (1) of Proposition 5.2.2.   □

We can now prove our main theorem of this chapter.

THEOREM 5.4.2. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional simple graded Lie algebra, and let ϕ be the natural embedding of $\mathfrak{g}$ into $\mathfrak{D} := \mathrm{Der}_K(S(\mathfrak{g}_{-1}^*))$. Let $\mathfrak{h}, \Delta, \Pi$, and β be as in the conclusion of Theorem 5.2.1, let W be the Weyl group generated by the reflections s_γ with $\gamma \in \Pi$, and let W_0 be the subgroup of W generated by the s_γ with $\gamma \in \Pi \setminus \{\beta\}$. Let ρ be the half sum of the positive roots of $\mathfrak{g}$, and let $\lambda - \rho$ be the highest root of $\mathfrak{g}$.*

Then $\mathfrak{D}$, regarded as a $\mathfrak{g}$-module through ϕ, has a finite composition chain, and each composition factor is isomorphic to $L(w(\lambda))$ for some $w \in W$. The multiplicity of the latter module among the composition factors is given by

$$(\mathfrak{D} : L(w(\lambda))) = \sum_{w_0 \in W_0} (-1)^{l(w_0)} P_{w_0, w}(1).$$

PROOF. By Propositions 5.4.1 and 5.2.2, the modules $\mathfrak{D}$ and $\mathfrak{D}^\vee = M_{\mathfrak{p}}(\mathfrak{g}_{-1})$ have the same composition factors. For a fixed irreducible module L in $\mathcal{O}$, the map sending $[M]$ to $(M : L)$ extends to a homomorphism from the Grothendieck group of $\mathcal{O}$ to $\mathbb{Z}$, so that

$$(\mathfrak{D} : L) = \sum_{w_0 \in W_0} (-1)^{l(w_0)} (M(w_0(\lambda)) : L)$$

by Proposition 5.3.1. By Proposition 5.3.2 and the BGG-criterion all composition factors of $\mathfrak{D}$ are of the form $L(w(\lambda))$ for some $w \in W$. The result now follows from the Kazhdan-Lusztig Conjecture, which is a theorem since [**2**] and [**12**]. $\qquad\square$

Evaluating the multiplicity formula of Theorem 5.4.2 on a computer for a fixed $w \in W$ can be rather time consuming. Therefore, it pays off to narrow down the set of $w \in W$ for which this has to be done.

LEMMA 5.4.3. *In the notation of Theorem 5.4.2, write $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$, and view $\Pi \backslash \{\beta\}$ as simple roots of $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0'$ in $\mathfrak{g}_0'$. Then $(\mathfrak{D} : L(w(\lambda))) > 0$ implies that $w(\lambda)$ restricts to a dominant weight on the Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0'$.*

PROOF. If the multiplicity of the $\mathfrak{g}$-module $L(w(\lambda))$ in $\mathfrak{D}$ is positive, then so is the multiplicity of the $\mathfrak{g}_0'$-module of highest weight $w(\lambda)|_{\mathfrak{h}_0}$. But $\mathfrak{D}$ is a direct sum of finite-dimensional $\mathfrak{g}_0'$-modules, hence $w(\lambda)|_{\mathfrak{h}_0}$ must be dominant. $\qquad\square$

From this lemma it follows that every right W_0-coset in W contains at most one element w with $(\mathfrak{D} : L(w(\lambda))) > 0$. Indeed, by the regularity of the action of W_0 on chambers in $\mathfrak{h}_0^*$, for any $w_1 \in W$, there is a unique $w_0 \in W_0$ such that $w_0(w_1(\lambda))$ is dominant. Using this observation for speedup, I wrote a program in LiE that takes a pair from the list at the end of page 74 as input, and determines the irreducible composition factors. Experiments with that program suggest the following conjecture.

CONJECTURE 5.4.4. *In the notation of Theorem 5.4.2, all irreducible quotients in a composition chain of $\mathfrak{D}$ have multiplicity 1. Define*
$$T := \{w \in W \mid (\mathfrak{D} : L(w(\lambda))) = 1\}.$$
Then the following holds: if $(\mathfrak{g}, \beta)$ is of type

(1) *(A_n, i) with $n \geq 1$ and $i \leq \frac{n}{2}$, then T consists of the elements*
$$w_j = (s_i s_{i+1} \cdots s_{i+j-1})(s_{i-1} s_i \cdots s_{i+j-2}) \cdots (s_{i-j+1} s_{i-j+2} \cdots s_i)$$
for $j = 0, \ldots, i$.

(2) *$(B_n, 1)$ with $n \geq 2$, then $T = \{e, s_1\}$.*

(3) *(C_n, n) with $n \geq 3$, then T consists of e and the elements*
$$w_j = (s_n)(s_{n-1} s_n)(s_{n-2} s_{n-1} s_n) \cdots (s_{n-2j} \cdots s_n)$$
for $j = 0, \ldots, \lceil \frac{n}{2} \rceil - 1$.

(4) *$(D_n, 1)$ with $n \geq 4$, then*
$$T = \{e, s_1, s_1 s_2 \cdots s_{n-2} s_n s_{n-1} s_{n-2} \cdots s_1\}.$$

(5) *(D_n, n) with $n \geq 4$, then T consists of e and the elements*
$$w_j = (s_n)(s_{n-2} s_{n-1})(s_{n-3} s_{n-2} s_n) \cdots$$
$$\cdots (s_{n-2j-2} s_{n-2j+3} \cdots s_{n-2} s_{n-1})(s_{n-2j-1} s_{n-2j+2} \cdots s_{n-2} s_n)$$
for $j = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$.

(6) *$(E_6, 1)$, then*
$$T = \{e, s_1, s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1\}.$$

(7) $(E_7, 7)$, *then*

$$T = \{e, s_7, s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7,$$

$$s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_4 s_2 s_6 s_5 s_4 s_3 s_1 s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7\}.$$

My program is not yet efficient enough to deal with the case $(E_7, 7)$; this is due to the fact that it is hard to compute Kazhdan-Lusztig polynomials for E_7. However, as Arjeh Cohen pointed out to me, the Weyl group elements w occurring above are all shortest representatives of their double cosets $W_0 w W_0$. This suggests the following conjecture, which predicts the above result for E_7.

CONJECTURE 5.4.5. *In the notation of Conjecture 5.4.4, every element w of T is the unique shortest element of its double coset $W_0 w W_0$. Moreover, if the Dynkin diagram of $\mathfrak{g}$ is simply laced, then the converse is also true.*

Remark 5.5.1 characterizes T of Conjecture 5.4.4 in the case where the diagram of $\mathfrak{g}$ is not simply laced.

EXAMPLE 5.4.6. If $\mathfrak{g}$ is of type $(A_3, 2)$ as in the second example of Section 5.1, then $\mathfrak{D}/\phi(\mathfrak{g})$ has a simple factor of highest weight $s_2 s_3 s_1 s_2(\lambda) - \rho$. With respect to the basis of fundamental weights, this weight equals $\mu := (1, -6, 1)$. In this example the center $\mathfrak{h}_c$ of $\mathfrak{g}_0$ is spanned by $H = H_1 + 2H_2 + H_3$, where H_1, H_2, H_3 is the Chevalley basis of $\mathfrak{h}$. The element H acts by the scalar $-2 * d$ on $\mathfrak{D}_d$, and by the scalar $\mu(H) = -10$ on the μ-weight space of $\mathfrak{D}$. This explains why $[\mathfrak{g}_1, \mathfrak{D}_4] \neq \mathfrak{D}_5$ as we noted in the introduction: the 4-dimensional $\mathfrak{g}_0'$-submodule of $\mathfrak{D}_5$ with highest weight $(1, 1) = \mu|_{\mathfrak{h}_0}$ is missing in $[\mathfrak{g}_1, \mathfrak{D}_4]$.

Conjectures 5.4.4 and 5.4.5 have been verified by computer for the cases where the rank of $\mathfrak{g}$ is at most 6. Moreover, the cases $(A_n, 1)$, $(B_n, 1)$ of Conjecture 5.4.4 can be proved as follows: in either case, one can write down the set S of all elements $w \in W$ with the property that $w(\lambda)$ is dominant on $\mathfrak{h}_0$; then $|S| = |W|/|W_0|$. For $w \in S \setminus \{e, s_1\}$, the Kazhdan-Lusztig polynomial $P_{e,w}$ turns out to have an easy form, and in fact to equal $P_{s_i,w}$ for some $i > 1$. But then the multiplicity $P_{e,w}(1)$ of $L(w(\lambda))$ in $M(\lambda)$ is equal to the multiplicity $P_{s_i,w}(1)$ of $L(w(\lambda))$ in $M(s_i(\lambda))$. By Propositions 5.4.1 and 5.3.2, this implies that $(\mathfrak{D} : L(w(\lambda))) = 0$. Finally, it is clear that $(\mathfrak{D} : L(\lambda)) = (\mathfrak{D} : L(s_1(\lambda))) = 1$.

This settles the conjecture in two special cases. However, a direct correspondence between irreducible composition factors of $\mathfrak{D}$ and double W_0-cosets of W, as suggested by Conjecture 5.4.5, would be more satisfactory. In many articles, the latter show up in connection with certain orbits on flag varieties; the next section reviews some of these results, and suggests how $\mathfrak{D}$ might be connected to this vast body of literature.

5.5. Towards a Correspondence

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a graded simple Lie algebra, let $\mathfrak{h}$, Π, and β be as in the conclusion of Theorem 5.2.1, and retain the notation Δ, $\Delta_\pm$, Π_0, Δ_0, $\Delta_{\pm 1}$, $\mathfrak{u}_\pm = \mathfrak{g}_{\mp 1}$, W, W_0 of Section 5.3.

Let G be any of the connected algebraic groups with Lie algebra $\mathfrak{g}$; it is simple over its center, so that the results of [52] apply. Define $\mathfrak{p}_\pm = \mathfrak{g}_{\mp 1} \oplus \mathfrak{g}_0$, and let $P_\pm$ be the closed connected subgroups of G with Lie algebras $\mathfrak{p}_\pm$, respectively. Now the exponential map defines an isomorphism of algebraic varieties from $\mathfrak{u}_+$ to the closed

connected subgroup U_+ of P_+ with Lie algebra $\mathfrak{u}_+$. The map $\iota : U_+ \to G/P_-$ sending u to uP_- is an open immersion, hence its image U_+P_-/P_- is an open neighbourhood of eP_- in G/P_-, and isomorphic to an affine space. By the construction of page 58, the action α of G on G/P_- can be differentiated to a homomorphism $\phi : X \mapsto -X*_\alpha$ from $\mathfrak{g}$ into $\operatorname{Der}_K K[U_+] = \operatorname{Der}_K S(\mathfrak{g}_{-1}^*) = \mathfrak{D}$; this is the required polynomial transitive realization.

Let G_0 be the closed connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Then G_0 acts on U_+ by conjugation and on G/P_- by left multiplication, and as $G_0 \subseteq P_-$, ι is a G_0-equivariant map between these. In [**52**] it is shown that

$$(16) \qquad\qquad \iota(G_0 u) = \iota(U_+) \cap (P_-\iota(u)) \text{ for all } u \in U_+.$$

Thus, ι induces a bijection between G_0-orbits on U_+ and P_--orbits on G/P_-. By the Bruhat lemma, the latter orbits are parameterized by the double cosets of W_0 in W. Richardson, Röhrle and Steinberg describe the G_0-orbits on U_+ more explicitly as follows. For $\alpha \in \Delta_{+1}$, denote by U_α the root subgroup of U_+ corresponding to α. Let $\beta_1, \ldots, \beta_l$ be a maximal sequence of mutually orthogonal long roots in Δ_{+1}, choose elements $u_i' \in U_{\beta_i} \setminus \{e\}$ for all i, and set

$$u_k := u_1' \cdots u_k'$$

for $k = 0, \ldots, l$. Then the orbits $C_k := G_0 u_k$ are all distinct and partition U_+, and the Zariski-closure $\overline{C_k}$ of C_k equals

$$\bigcup_{0 \le k' \le k} C_{k'};$$

in particular, C_l is an open dense orbit of U_+. Similarly, set $w_k := s_{\beta_1} \cdots s_{\beta_k}$ for $k = 0, \ldots, l$. Then $\{w_1, \ldots, w_l\}$ is a system of representatives for the double cosets of W_0 in W. Finally, if we represent each w_k by an element $\tilde{w}_k$ of the normalizer of the Cartan subgroup of G with Lie algebra $\mathfrak{h}$, then the set $\{\tilde{w}_1 P_-, \ldots, \tilde{w}_l P_-\}$ is a system of representatives for the P_--orbits on G/P_-.

REMARK 5.5.1. For $\mathfrak{g}$ of type (C_n, n) or $(B_n, 1)$, the set T of Conjecture 5.4.4 consists of shortest representatives of the double cosets $W_0 w_k W_0$ with k zero or odd, i.e., approximately half of all double cosets are represented by T.

We proceed with some results from [**56**]. For $k = 0, \ldots, l$, let $I_k \subseteq K[U_+]$ be the defining ideal of $\overline{C_k}$. These ideals form a chain

$$K[U_+] =: I_{-1} \supset I_0 \supset I_1 \supset \ldots \supset I_l = 0.$$

By (16), $\iota(\overline{C_k})$ is the intersection of a closed P_--stable subset of G/P_- and the open dense subset $\iota(U_+)$ of G/P_-; from this it follows that I_k is invariant under $\phi(\mathfrak{p}_-) = \phi(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$. Tanisaki shows that, moreover, each I_k is generated by its homogeneous part $(I_k)_{k+1}$ of degree $k + 1$, and that

$$\phi(\mathfrak{g}_{-1})(I_k)_{k+1} = (I_{k-1})_k$$

for all $k = 0, \ldots, l - 1$. Note that this implies $\mathfrak{D}(I_k) = I_{k-1}$ for all $k = 0, \ldots, l - 1$.

It seems natural to think that the chain $I_0, \ldots, I_l$ of ideals, having the right cardinality and satisfying the invariance properties mentioned above, should lead to a chain of $\mathfrak{g}$-submodules of $\mathfrak{D}$ that would prove Conjecture 5.4.5. By way of example, consider

once again the case where $\mathfrak{g}$ is of type $(A_3, 2)$. Take $G = \mathrm{SL}_4$, and P_-, N_+ and G_0 with the following decompositions into (2×2)-blocks:

$$P_- = \left\{ \begin{pmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{pmatrix} \mid \det(p_{11}) \det(p_{22}) = 1 \right\}, \ N_+ = \left\{ \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \right\}, \ \text{and}$$

$$G_0 = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid \det(g_1) \det(g_2) = 1 \right\}.$$

Denote by g_0 and n_+ the typical elements of G_0 and n_+ above. Then we have

$$g_0 n_+ g_0^{-1} = \begin{pmatrix} I & g_1 x g_2^{-1} \\ 0 & I \end{pmatrix},$$

from which it follows that the G_0-orbits on N_+ are C_0, C_1, and C_2, where $C_i = \{n_+ \in N \mid \mathrm{rk}\, x = i\}$. Writing x_{ij} for the entries of c, the defining ideals of $\overline{C_0}, \overline{C_1}$, and $\overline{C_2}$ are $I_0 = (x_{11}, x_{12}, x_{21}, x_{22})$, $I_1 = (x_{11}x_{22} - x_{12}x_{21})$, and $I_2 = 0$, respectively. To calculate the action of $\mathfrak{g}$ on $K[(x_{ij})_{ij}]$, let X_{ij} $(i, j = 1, 2)$ be (2×2)-matrices over K with $\mathrm{tr}(X_{11} + X_{22}) = 0$, and compute over the algebra $A := K[\epsilon]/(\epsilon^2)$ (recall that $P_-(A)$ denotes the group of points of P_- with coordinates in A; see [**5**], §1 nr. 5):

$$\begin{pmatrix} I - \epsilon X_{11} & -\epsilon X_{12} \\ -\epsilon X_{21} & I - \epsilon X_{22} \end{pmatrix} \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} P_-(A)$$

$$= \begin{pmatrix} I - \epsilon X_{11} & x - \epsilon(X_{11}x + X_{12}) \\ -\epsilon X_{21} & I - \epsilon(X_{21}x + X_{22}) \end{pmatrix}$$

$$\cdot \begin{pmatrix} I - \epsilon(X_{21}x + X_{22}) & 0 \\ 0 & I + \epsilon(X_{21}x + X_{22}) \end{pmatrix} P_-(A)$$

$$= \begin{pmatrix} I - \epsilon(X_{11} + X_{21}x + X_{22}) & x + \epsilon(xX_{21}x + xX_{22} - X_{11}x - X_{12}) \\ -\epsilon X_{21} & I \end{pmatrix}$$

$$\cdot \begin{pmatrix} I & 0 \\ -\epsilon X_{21} & I \end{pmatrix} P_-(A)$$

$$= \begin{pmatrix} I - \epsilon(X_{11} + X_{22} + [X_{21}, x]) & x + \epsilon(xX_{21}x + xX_{22} - X_{11}x - X_{12}) \\ 0 & I \end{pmatrix}$$

$$\cdot \begin{pmatrix} I + \epsilon(X_{11} + X_{22} + [X_{21}, x]) & 0 \\ 0 & I \end{pmatrix} P_-(A)$$

$$= \begin{pmatrix} I & x + \epsilon(xX_{21}x + xX_{22} - X_{11}x - X_{12}) \\ 0 & I \end{pmatrix} P_-(A),$$

where the penultimate equality is justified by $\mathrm{tr}(X_{11} + X_{22} + [X_{21}, x]) = 0$. Hence, the realization ϕ is given explicitly by

$$\phi \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \sum_{i,j \in \{1,2\}} (xX_{21}x + xX_{22} - X_{11}x - X_{12})_{ij} \partial_{ij},$$

where ∂_{ij} denotes differentiation with respect to x_{ij}. Note that $\mathfrak{g}_i$ is mapped into $\mathfrak{D}_i$, as expected. Let E_{ij} denote the matrix with a 1 on position (i, j) and zeroes elsewhere. Then for example

$$\phi(E_{32}) = x_{11}(x_{21}\partial_{11} + x_{22}\partial_{12}) + x_{21}(x_{21}\partial_{21} + x_{22}\partial_{22}),$$

and
$$\phi(E_{32})(x_{11}x_{22} - x_{12}x_{21}) = x_{21}(x_{11}x_{22} - x_{12}x_{21}),$$
so that $\phi(E_{32})(I_1) \subseteq I_1$, as expected.

We have seen that in this case the subspace $\bigoplus_{d=-1}^{4} \mathfrak{D}_d$ generates a proper $\mathfrak{g}$-submodule M of $\mathfrak{D}$, and this is the only 'non-obvious' submodule. Similarly, I_1 is the only 'non-obvious' ideal among the I_k. Hence we conclude that the search for a direct correspondence between G_0-orbits on U_+ and the irreducible composition factors of the $\mathfrak{g}$-module $\mathfrak{D}$ suggests the following challenging starting point for further research: describe M in terms of I_1!

Transitive Lie Algebras in One and Two Variables

This appendix briefly discusses Lie's classification of transitive Lie algebras in one and two variables over an algebraically closed field K of characteristic zero. They correspond to effective pairs of codimension one or two.

There are three classes of transitive Lie algebras in one variable; using Lie's notation $x := x_1$ and $p := \partial_x$ they are

$$\langle p \rangle, \langle p, xp \rangle, \text{ and } \langle p, 2xp, -x^2 p \rangle.$$

A proof of this classification can be found in [**47**]. Also, it is an easy consequence of the classification of primitive pairs; see [**21**] for a short proof for the case where $\mathfrak{g}$ is not simple.

Effective Pairs of Codimension Two. We write $x := x_1, y := x_2, p := \partial_1$, and $q := \partial_2$ in transitive Lie algebras in two variables. The primitive Lie algebras are listed in table 1. The third column of table 1 contains the the label given to these algebras in [**22**] and [**41**].

Now consider an effective pair $(\mathfrak{g}, \mathfrak{k})$ of codimension two which is not primitive. Then there exists a subalgebra $\mathfrak{k}_1$ such that $\mathfrak{g} \supset \mathfrak{k}_1 \supset \mathfrak{k}$, where the inclusions are proper. In general, the pairs $(\mathfrak{g}, \mathfrak{k}_1)$ and $(\mathfrak{k}_1, \mathfrak{k})$ will no longer be effective. Denote the largest $\mathfrak{g}$-ideal contained in $\mathfrak{k}_1$ by $\mathfrak{i}$ and the largest $\mathfrak{k}_1$-ideal contained in $\mathfrak{k}$ by $\mathfrak{j}$. The quotients $(\mathfrak{g}/\mathfrak{i}, \mathfrak{k}_1/\mathfrak{i})$ and $(\mathfrak{k}_1/\mathfrak{j}, \mathfrak{k}/\mathfrak{j})$ are both effective pairs of codimension one, and hence there are nine possibilities for the pair $(\dim \mathfrak{g}/\mathfrak{i}, \dim \mathfrak{k}_1/\mathfrak{j})$. This pair is called the *type* of the triple $(\mathfrak{g}, \mathfrak{k}_1, \mathfrak{k})$. In our table 2, $\mathfrak{k}_1$ is suppressed. Sometimes, various intermediate subalgebras $\mathfrak{k}_1$ can be chosen, and the resulting triples may be of different types. As a consequence, an effective pair $(\mathfrak{g}, \mathfrak{k})$ may occur several times in the classification. When this happens, there will be a reference to an earlier entry of the list, to which it is isomorphic.

Table 2 differs from Lie's table in that the origin $(0,0)$ is always a regular point. Moreover, whenever an $\mathfrak{sl}_2$ occurs, its Chevalley basis is contained in the basis given in that table. The third column can be used for translation between this table and Lie's table in [**41**].

Type	Realization	
(5)	$\langle p, q, xq, xp - yq, yp \rangle$	A3
(6)	$\langle p, q, xq, xp - yq, yp, xp + yq \rangle$	A2
(8)	$\langle p, q, xq, xp - yq, yp, xp + yq, x^2 + xyq, xyp + y^2 q \rangle$	A1

TABLE 1. The primitive Lie algebras in two variables.

Type	Case	Realization	Label
(1,1)		$\langle p, x^i \exp(\alpha x)q\rangle$,	Bβ1,D1,D2
		where $i = 0, \ldots, r_\alpha \geq 0$	
		and α in a non-empty finite set	
(1,2)		$\langle p, yq, x^i \exp(\alpha x)q\rangle$,	Bβ2,C2
		where $i = 0, \ldots, r_\alpha \geq 0$	
		and α in a non-empty finite set	
(1,3)		$\langle p, q, 2yq, -y^2q\rangle$	C5
(2,1)	1	$\langle p, xp + q\rangle$	
		For a different $\mathfrak{k}_1$, this is (1,1).	
(2,1)	2	$\langle p, xp, x^iq\rangle$, where $i = 0, \ldots, r \geq 0$	Bγ1
		For $r = 0$ and a different $\mathfrak{k}_1$, this is (1,2).	
(2,2)	1	$\langle p, xp - \lambda yq, x^iq\rangle$, where $i = 0, \ldots, r \geq 0$ and $\lambda \neq 0$	Bγ2,C8,D3
(2,2)	2	$\langle p, xp + ((r+1)y + x^{r+1})q, x^iq\rangle$,	Bγ3
		where $i = 0, \ldots, r \geq 0$	
(2,2)	3	$\langle p, xp, yq, x^iq\rangle$, where $i = 0, \ldots, r \geq 0$	Bγ4,C3
(2,3)		$\langle p, xp, q, 2yq, -y^2q\rangle$	C6
(3,1)	1	$\langle p, 2xp + q, -x^2p - xq\rangle$	Bδ1
(3,1)	2	$\langle p, 2xp, -x^2p, q\rangle$	
		For a different $\mathfrak{k}_1$, this is (1,3).	
(3,1)	3	$\langle p, 2xp - q, -x^2p + xq, q\rangle$	Bδ2
(3,2)	1	$\langle p, 2xp - 2yq, -x^2p + (1 + 2xy)q\rangle$	C9
(3,2)	2	$\langle p, 2xp + ryq, -x^2p - rxyq, x^iq\rangle$,	Bδ3
		where $i = 0, \ldots, r \geq 1$	
(3,2)	3	$\langle p, 2xp + ryq, -x^2p - rxyq, yq, x^iq\rangle$,	Bδ4
		where $i = 0, \ldots, r \geq 0$	
		For $r = 0$ and a different $\mathfrak{k}_1$ this is (2,3).	
(3,3)		$\langle p, 2xp, -x^2p, q, 2yq, -y^2q\rangle$	C7

TABLE 2. The non-primitive transitive Lie algebras in two variables.

An Implementation of the Realization Formula

My research for Chapters 2 and 5 benefited greatly from my implementation of the Realization Formula of Chapter 2 in the computer algebra system `GAP` (release 4.2) [**20**]. Although everyone in this field knows that it is, in principle, possible to compute explicitly a realization of any abstractly given pair $(\mathfrak{g}, \mathfrak{k})$, many people were surprised by the simplicity of the Realization Formula. Therefore, I discuss some details of the implementation, and hope that others will find my program useful. I refer to my web page `www.win.tue.nl/~jdrai` for the complete implementation.

B.1. Design of the Algorithm

Recall the setting of the Realization Formula: we are given a finite-dimensional Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{k}$ of codimension n, and elements $Y_1, \ldots, Y_n$ of $\mathfrak{g}$ that project onto a basis of $\mathfrak{g}/\mathfrak{k}$; and we are to compute the realization $\phi_{\mathbf{Y}}$. This linear map is determined by the list $(\phi_{\mathbf{Y}}(X_1), \ldots, \phi_{\mathbf{Y}}(X_k), \phi_{\mathbf{Y}}(Y_1), \ldots, \phi_{\mathbf{Y}}(Y_n))$, where $X_1, \ldots, X_k$ form a basis of $\mathfrak{k}$. In general, $\phi_{\mathbf{Y}}$ has formal power series coefficients, and we have to specify the degree d up to which these coefficients must be calculated. We will therefore derive a function `Blattner` that takes as input $\mathfrak{g}, (\mathbf{X}, \mathbf{Y}) =: \mathbf{Z}, n$, and d, and that produces the above list of images, truncated at degree d.

`GAP` is well suited for implementing `Blattner`, because its standard distribution already has many functions dealing with Lie algebras [**24**], [**25**]. In particular, `GAP` has a function called `UniversalEnvelopingAlgebra` that takes $\mathfrak{g}$ as input and returns the universal enveloping algebra $U(\mathfrak{g})$. The elements of $U(\mathfrak{g})$ are written as linear combinations of the PBW-basis corresponding to a basis $\mathbf{Z}$ of $\mathfrak{g}$. Unfortunately, in release 4.2, `UniversalEnvelopingAlgebra` takes for $\mathbf{Z}$ some default basis of $\mathfrak{g}$, rather than accepting $\mathbf{Z}$ as a parameter. As we saw in Chapter 2, the realization $\phi_{\mathbf{Y}}$ depends heavily on $\mathbf{Y}$, so that we need a slightly adapted version of `UniversalEnvelopingAlgebra`. Willem de Graaf kindly provided me with such a function, `UEA`, with which $\phi_{\mathbf{Y}}$ is easily implemented.

It remains to decide on the form of the output of `Blattner`. To be able to use that output for further calculations, it is convenient to construct the Lie algebra $\mathfrak{D}$ of polynomial vector fields in `GAP` first, and to let `Blattner` return a list of elements of $\mathfrak{D}$. To minimize our programming effort, we view $\mathfrak{D}$ as a Lie subalgebra of the associative Weyl algebra W with generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and relations

$$[x_i, x_j] = 0, \ [\partial_i, x_j] = \delta_{ij}, \ \text{and} \ [\partial_i, \partial_j] = 0$$

for all $i, j = 1, \ldots, n$. The algebra W, in turn, is isomorphic to the quotient $U(\mathfrak{l})/(I-1)$, where $\mathfrak{l}$ is the $(2n+1)$-dimensional Lie algebra with basis $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, I$ and

Lie bracket determined by

$$[x_i, x_j] = 0, \ [\partial_i, x_j] = \delta_{ij} I, \ [\partial_i, \partial_j] = 0,$$
$$[x_i, I] = 0, \text{ and } [\partial_i, I] = 0$$

for all $i, j = 1, \ldots, n$. The Lie algebra $\mathfrak{l}$ is easily constructed using the function `LieAlgebraByStructureConstants`, and another slight modification of the function `UniversalEnvelopingAlgebra` allows us to construct $U(\mathfrak{l})/(I - 1)$. This leads to a function `WeylAlgebra` to be called with two arguments: the ground field and the number n of variables.

This finishes the description of the non-standard functions `UEA` and `WeylAlgebra` that appear in the source code for `Blattner`. That source code, which forms the next section, is self-evident. I include one example of usage.

EXAMPLE 2.1.1. The Lie algebra $\mathfrak{g}$ of Example 2.4.3 can be entered into `GAP` by its structure constants.

```
gap> T:=EmptySCTable(7,0,"antisymmetric");;
gap> SetEntrySCTable(T,2,1,[2,1]);
gap> SetEntrySCTable(T,2,3,[-2,3]);
gap> SetEntrySCTable(T,1,3,[1,2]);
gap> SetEntrySCTable(T,2,5,[1,5]);
gap> SetEntrySCTable(T,2,6,[-1,6]);
gap> SetEntrySCTable(T,3,5,[1,6]);
gap> SetEntrySCTable(T,1,6,[1,5]);
gap> SetEntrySCTable(T,4,5,[1,5]);
gap> SetEntrySCTable(T,4,6,[1,6]);
gap> SetEntrySCTable(T,4,7,[2,7]);
gap> SetEntrySCTable(T,5,6,[1,7]);
gap> g:=LieAlgebraByStructureConstants(Rationals,T);;
```

Here the basis vectors E, H, F, I, X, Y, Z are numbered $1, \ldots, 7$, respectively. The realization $\phi_{(I,X,Y,Z)}$ of Theorem 2.4.2 can now be computed as follows.

```
gap> B:=Basis(g);;
gap> Blattner(g,B,4,4)[1];
[ [(-1)*x_3*D_2+(1/2)*x_3^2*D_4],
  [(-1)*x_2*D_2+(1)*x_3*D_3],
  [(-1)*x_2*D_3+(1/2)*x_2^2*D_4],
  [(-1)*x_2*D_2+(-1)*x_3*D_3+(-2)*x_4*D_4+( 1)*D_1],
  [(-1)*x_3*D_4+(1)*D_2],
  [(1)*D_3],
  [(1)*D_4] ]
```

B.2. Source Code of `Blattner`

```
Constant:=function(n,e)
#This function returns a vector all of whose n entries equal e.
  return List([1..n], a->e);
end;
```

```
ExponentsOfGivenDegree:=function(n,d)
#This function returns all multi-indices with n entries and total
#degree d.
  return OrderedPartitions(d+n,n)-Constant(n,1);
end;


Power:=function(a,m,one)
#This function computes the product (a_1^m_1)*(a_2^m_2)*...; both
#lists should be equally long. The a_i are assumed to be elements
#of a monoid, the identity of which is the third parameter. The m_i
#must be natural numbers.
local i,r;
  r:=one;
  for i in [1..Length(m)] do
    if m[i]<>0 then r:=r*a[i]^m[i]; fi;
  od;
  return(r);
end;


CoefficientsOfLinearPart:=function(u,l)
#Here, u is an element of a Universal Enveloping Algebra of a Lie
#algebra of dimension at least l. The function returns, from the
#linear part of u, only the variables with index>l, with their
#coefficients. Hence, for l=6 and
#u=[(-1)*x.1*x.7+(1)*x.3*x.7*x.8+(1)*x.6*x.8
#  +(5)*x.6+(-1)*x.7+(2)*x.8]
#the list [[7,-1],[8,2]] is returned.
local ret, e, i;
  ret:=[];
  e:=ExtRepOfObj(u)[2];
  for i in [1..Length(e)/2] do
    if (Length(e[2*i-1])=2) and (e[2*i-1][2]=1) and
        (e[2*i-1][1]>l)
      then AddSet(ret,[e[2*i-1][1],e[2*i]]);
    fi;
  od;
  return(ret);
end;

Blattner:=function(g,Z,n,d)
#The parameters are:
#g:  a Lie algebra
#Z:  a linear basis of g.
#n:  The elements Z[1], Z[2],...,Z[Length(Z)-n] are supposed
#  to span a subalgebra k.
#d:  a natural number
```

```
#This function computes a realization of the pair (g,k). More
#precisely, it returns a pair (L,W), where W is the Weyl algebra
#in n variables, and L, whose entries are elements of W, is the
#image of Z under the realization, with coefficients truncated
#after degree d.

local k,K,U,W,one,X,Y,x,D,L,e,exps,
Ymons,xmons,facs,i,j,coeffs,t;
  k:=Dimension(g)-n;
    #This is the dimension of k.
  K:=LeftActingDomain(g);
    #The field.

  U:=UEA(g,Z);
    #The universal enveloping algebra of g with
    #PBW-basis corresponding to Z.

  one:=Identity(U);
  X:=GeneratorsOfAlgebraWithOne(U);
  Y:=X{[k+1..k+n]};
    #These form a basis of g complementary to k if
    #we identify both Lie algebras with subspaces
    #of U.

  W:=WeylAlgebra(K,n);
  x:=GeneratorsOfAlgebraWithOne(W){[1..n]};
  D:=GeneratorsOfAlgebraWithOne(W){[n+1..2*n]};
    #These will be used to return the result.

  L:=Constant(k+n,Zero(W));
    #The realization will be stored in L.

  for e in [0..d] do
    exps:=ExponentsOfGivenDegree(n,e);
    Ymons:=List(exps,m->Power(Y,m,one));
      #These are the monomials Y^m in U
    xmons:=List(exps,m->Power(x,m,Identity(W)));
      #These are the monomials x^m
    facs:=List(exps,m->Product(List(m,Factorial)));
      #This is a list of factorials m!
    for i in [1..Length(exps)] do
      for j in [1..k+n] do
        coeffs:=
        CoefficientsOfLinearPart(Ymons[i]*X[j],k);
      #This computes the relevant linear part
      #of Y^m X[j]
```

```
      for t in coeffs do
        L[j]:=L[j]+
        (t[2]/facs[i])*xmons[i]*D[t[1]-k];
      od;
    od;
  od;
od;
return [L,W];
end;
```

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Samenvatting

Intuïtief schrijft een vectorveld X in elk punt p op een oppervlak M een richting $X|_p$ voor. Het vanuit een gegeven beginpunt op M volgen van het vectorveld heet *integratie* van dat vectorveld. Met behulp van integratie kan de *commutator* van twee vectorvelden worden geconstrueerd, zoals de omslag van dit proefschrift illustreert. Uitgerust met deze commutator is de lineaire ruimte van alle vectorvelden op M een Lie-algebra, of eigenlijk mogen we wel zeggen: dè Lie-algebra, want dit is de algebra waarvan Sophus Lie de eindig-dimensionale deelalgebra's bestudeerde.

Als f een in zekere zin mooie functie op M is, dan kan men in elk punt p op M de afgeleide $X|_p(f)$ van f in de richting $X|_p$ bepalen; laat $X(f)$ de functie op M zijn die p afbeeldt op $X|_p(f)$. Idealiter is $X(f)$ wederom een mooie functie op M, en is de afbeelding $f \mapsto X(f)$ een *derivatie* op mooie functies op M, dat wil zeggen een lineaire afbeelding die bovendien aan Leibniz' identiteit $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$ voldoet. Onder de juiste voorwaarden wordt X vastgelegd door deze afbeelding, en hebben we vectorvelden op M vertaald in derivaties op de commutatieve algebra van mooie functies op M.

Dit stelt ons in staat het intuïtieve beeld van vectorvelden te verlaten en onze aandacht te richten op commutatieve algebra's zoals $K[[x_1, \ldots, x_n]]$ (formele macht-reeksen in $x_1, \ldots, x_n$ met coëfficiënten uit het lichaam K), $K[x_1, \ldots, x_n]$ (polynomen in die variabelen) of $K[U]$: de algebra van reguliere functies op een affiene algebraïsche variëteit U over K; hier speelt U dus de rol van M uit de eerste twee paragrafen. Hierbij nemen we altijd aan dat de karakteristiek van K nul is. Als A zo'n algebra is, dan schrijven we $\mathrm{Der}_K(A)$ voor de ruimte van K-lineaire derivaties op A. De op de omslag geïllustreerde 'meetkundige' commutator correspondeert met de 'gewone' commutator $[X, Y] := X \circ Y - Y \circ X$, en inderdaad is $\mathrm{Der}_K(A)$, uitgerust met deze commutator, een Lie-algebra.

In navolging van Lie en in analogie met het groepentheoretische begrip, noemen we een deelalgebra $\mathfrak{l}$ van de Lie-algebra $\hat{\mathfrak{D}}^{(n)} := \mathrm{Der}_K(K[[x_1, \ldots, x_n]])$ *transitief*, wanneer $\mathfrak{l}$ voor elke $i = 1, \ldots, n$ een element bevat van de vorm

$$\partial_i + \text{een vectorveld dat in 0 verdwijnt.}$$

De verzameling van transitieve deelalgebra's van $\hat{\mathfrak{D}}^{(n)}$ valt uiteen in klassen, bestaande uit algebra's die door formele coördinatentransformaties in elkaar worden overgevoerd. Volgens de realisatiestelling van Guillemin en Sternberg corresponderen deze klassen precies met de isomorfieklassen van paren $(\mathfrak{g}, \mathfrak{k})$, waarbij $\mathfrak{g}$ een Lie-algebra over K is, en $\mathfrak{k}$ een deelalgebra van $\mathfrak{g}$ die codimensie n in $\mathfrak{g}$ heeft en geen niet-triviale $\mathfrak{g}$-idealen bevat. Blattners constructieve bewijs van deze stelling ligt ten grondslag aan de expliciete Realisatieformule die in Hoofdstuk 2 wordt afgeleid.

Het bestaan van realisaties met formele coefficiënten, door de Realisatieformule concreet gemaakt, roept de vraag op of we met bescheidener coefficiënten toe kunnen; deze vraag vormt een rode draad door dit proefschrift. Een eerste vermoeden ten aanzien van die coefficiënten werd al geopperd door Sophus Lie. Voor $n = 1, 2$ beschrijft Lie alle klassen van *eindig-dimensionale* transitieve deelalgebra's van $\hat{\mathfrak{D}}^{(n)}$. Hij claimt de classificatie voor $n = 3$ ook te hebben voltooid, maar publiceerde slechts een deel, omdat de volledige lijst hem te lang was. Hij merkt echter op dat voor $n = 3$, net als voor $n = 1, 2$, elke eindig-dimensionale transitieve Lie-algebra door een coördinatentransformatie op zodanige vorm te brengen is, dat de coefficiënten der ∂_i bepaalde eenvoudige functies zijn; hij spreekt bovendien het vermoeden uit dat dit voor grotere n ook zo is. In Hoofdstuk 2 worden enkele stellingen bewezen ten gunste van dit vermoeden. In het bijzonder bewijzen we Lies vermoeden voor *zeer imprimitieve paren*, dat wil zeggen eindig-dimensionale paren $(\mathfrak{g}, \mathfrak{k})$ waarvoor er een keten

$$\mathfrak{g} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \ldots \supset \mathfrak{g}_0 = \mathfrak{k}$$

bestaat, waarin $\mathfrak{g}_i$ een $(\dim \mathfrak{k} + i)$-dimensionale deelalgebra van $\mathfrak{g}$ is. Verrassend genoeg zijn dit voor $n = 3$ precies de paren zijn waarvan Lie zijn classificatie niet gepubliceerd heeft.

Aan het andere eind van het scala aan eindig-dimensionale transitieve Lie-algebra's bevinden zich de *primitieve* Lie-algebra's, die corresponderen met paren $(\mathfrak{g}, \mathfrak{k})$ waarvoor $\mathfrak{k}$ een maximale deelalgebra van $\mathfrak{g}$ is. Over een algebraïsch gesloten lichaam zijn deze geclassificeerd door Morozov en Dynkin. Voorbeelden zijn de *simpel-parabolische* paren, waarvoor $\mathfrak{g}$ simpel is en $\mathfrak{k}$ een maximale parabolische deelalgebra; deze paren hebben een realisatie met polynomiale coefficiënten. Sommige andere primitieve paren laten zich op een natuurlijke manier inbedden in simpel-parabolische paren, en erven daarvan een realisatie met polynomiale coefficiënten. Deze kunnen dus worden afgeschreven als kandidaat-tegenvoorbeelden voor Lies vermoeden. Dit motiveert de zoektocht in Hoofdstuk 3 naar inbeddingen van primitieve in simpel-parabolische paren. Eerst bewijzen we dat een simpel-parabolisch paar zelf bijna nooit een echt deelpaar is van een ander paar en vervolgens tonen we aan dat de meeste primitieve paren zich niet laten inbedden in simpel-parabolische paren. Niettemin zijn er uitzonderingen op deze uitspraken, en die worden alle beschreven.

Uit de classificaties van Morozov en Dynkin volgt dat alle primitieve paren $(\mathfrak{g}, \mathfrak{k})$ *algebraïsch* zijn in de zin dat $\mathfrak{g}$ de Lie-algebra is van een affiene algebraïsche groep G, en $\mathfrak{k}$ de Lie-algebra van een Zariski-gesloten ondergroep H. In dit geval induceert de actie van G op de homogene ruimte G/H (die de rol van M uit de eerste twee paragrafen van deze samenvatting speelt) een actie van $\mathfrak{g}$ door middel van vectorvelden op de schoof van reguliere functies op G/H; in het bijzonder vinden we een homomorfisme $\mathfrak{g} \to \mathrm{Der}_K(K[U])$ voor elke affiene open deelverzameling U van G/H. De theorie van zulke homogene ruimten kan nu worden aangewend om voor nog meer algebraïsche paren het bestaan van een realisatie met polynomiale coefficiënten aan te tonen. Deze aanpak wordt behandeld in Hoofdstuk 4. Het grootste deel van dat hoofdstuk wordt echter besteed aan de omgekeerde vraag: stel we hebben een homomorfisme $\mathfrak{g} \to \mathrm{Der}_K(K[U])$ van Lie-algebra's; laat dit homomorfisme zich integreren tot een actie van een algebraïsche groep? Het eerste hoofdresultaat van dat hoofdstuk is van toepassing als de

actie van $\mathfrak{g}$ op $K[U]$ lokaal eindig is; het tweede is algemeen van toepassing, maar minder krachtig omdat vooraf wordt aangenomen dat $\mathfrak{g}$ de Lie-algebra is van een affiene algebraïsche groep.

Zoals in Hoofdstuk 3 wordt aangetoond, is bijna elk primitief simpel-parabolisch paar $(\mathfrak{g}, \mathfrak{p})$ maximaal, waaruit volgt dat zijn beeld onder een polynomiale transitieve realisatie ϕ een maximale eindig-dimensionale deelalgebra is van de Lie-algebra $\mathfrak{D} := \mathrm{Der}_K(K[x_1, \ldots, x_n])$, waar $n = \mathrm{codim}_{\mathfrak{g}} \mathfrak{p}$. Dit leidt tot de subtielere vraag: is $\mathfrak{D}/\phi(\mathfrak{g})$ zelfs een irreducibel $\mathfrak{g}$-moduul? Deze vraag wordt in hoofdstuk 5 bestudeerd voor het geval dat $\mathfrak{p}$ een Abels nilpotent radicaal heeft. We bewijzen dat $\mathfrak{D}$ een eindige compositieketen heeft als $\mathfrak{g}$-moduul, leiden een formule af voor de multipliciteiten van irreducibele factoren in zo'n compositieketen, en formuleren een op computerexperimenten met deze formule gebaseerd vermoeden dat deze multipliciteiten beschrijft.

Bij lezing van dit proefschrift zal men vaststellen dat elk hoofdstuk meer vragen oproept dan het beantwoordt. Zo blijft Lies vermoeden open in Hoofdstuk 2, komen inbeddingen tussen twee primitieve paren die beide niet simpel-parabolisch zijn niet aan de orde in Hoofdstuk 3, eindigt Hoofdstuk 4 met de vraag of de conditie—in het tweede hoofdresultaat van dat hoofdstuk—dat $\mathfrak{g}$ de Lie-algebra is van een algebraïsche groep, niet afgezwakt of weggelaten kan worden, en blijft de relatie tussen bepaalde dubbele nevenklassen in een Weyl-groep en de irreducibele quotiënten in de compositieketen van $\mathfrak{D}$ als $\mathfrak{g}$-moduul onopgehelderd in Hoofdstuk 5. Elk van deze vragen vormt een mooi uitgangspunt voor verder onderzoek.

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Eindhoven, april 2002   Jan Draisma

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As of September 2002: Department of Mathematics, Universität Basel, Switzerland.

Education

June 2002, planned: Ph.D. in Mathematics, Technische Universiteit Eindhoven (TU/e).

September 1998: Master's degree in Applied Mathematics, TU/e. Thesis: 'An algebraic Approach to Lie Symmetries of Ordinary Differential Equations', supervisor: Arjeh M. Cohen.

July 1993: Graduation from secondary school, Dr. Schaepmancollege, Dongen.

Academic Goals

Being fond of mathematics in general, and of (computer) algebra in particular, I wish to continue doing research in areas where computer algebra can play the role of laboratory in our constant effort towards a better understanding of the intriguing beauty of mathematical concepts.