Some tropical geometry of algebraic groups, minimal orbits, and secant varieties

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1. Secant varieties and minimal orbits

Given a variety X embedded in a projective space $\mathbb{P}V$, the $(k-1)$-st secant variety of X, denoted kX, is the closure of the union of all $(k-1)$-spaces spanned by k points on X. We usually require that X spans $\mathbb{P}V$, so that $kX = \mathbb{P}V$ for k sufficiently large. We often work with the cone C in V over X rather than with X, and write kC for the cone over kX. Secant varieties appear in applications as diverse as phylogenetics [2, 5, 12], complexity theory [10, 11], and polynomial interpolation [1]. The references in this note are by no means complete, but they themselves contain many further relevant references.

Example 1.1. Consider "matrix multiplication of two 2×2-matrices", which can be thought of as a tensor T in $V = \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, and take C equal to the set of pure tensors in this tensor product. Then the ordinary procedure for multiplication, which needs 8 multiplications, shows that T lies in $8C$. Strassen realised that by taking clever linear combinations, T can be written as a sum of 7 pure tensors. This shows that $T \in 7C$, and Strassen used this fact in an algorithm for multiplication of $n \times n$-matrices which needs less than n^3 multiplications [11]. Recently Landsberg proved that $T \notin 6C$ [10]—which means that T cannot be approximated with tensors of rank 6, a much stronger and more difficult statement than that T itself does not have rank 6.

Example 1.2. In phylogenetics, one tries to reconstruct evolution from genetic data of species alive today. One approach runs as follows: given n strings of nucleotides A, C, G, T of DNA of n species and given a hypothetical evolutionary tree leading to those n species, one wants to decide whether the tree matches the data. First, the data leads to an empirical probability distribution on $\{A, C, G, T\}^n$, which can be thought of as an element of $(\mathbb{C}^4)^{\otimes n}$. On the other hand one has a parameterised variety, the *General Markov Model*, of probability distributions that match the tree. To test whether the tree matches the data, one tries to find the equations defining the model, which can then be tested on the empirical distribution. Allman and Rhodes reduced the quest for equations defining the model for general trees to the case of *stars*, trees of diameter at most 2 [2]. For the star with 3 leaves, this model is $4C$, where C is the set of pure tensors in $(\mathbb{C}^4)^{\otimes 4}$—for which we unfortunately do not know equations yet. On a side note, Allman and Rhodes prove only that their procedure would yield set-theoretic equations; we recently showed that they generate the full ideal.

Theorem 1.3 ([9])**.** *The Allman-Rhodes equations generate the full ideal of the phylogenetic model.*

Example 1.4. The case where $V = S^d(\mathbb{C}^n)$ and C is the set of pure powers l^d with $l \in \mathbb{C}^n$ is closely related to polynomial interpolation. The dimensions of the secant varieties kC are known from the ground-breaking work of Alexander and Hirschowitz [1].

This illustrates the omnipresence of secant varieties in mathematics and applications. Two important problems concerning them are: first, to find equations for kC; and second, more modestly, to determine the dimension of kC. We now concentrate on the second problem. Typically one expects $\dim kC$ to be $\min\{k \dim C, \dim V\}$—an obvious upper bound—but one has a hard time proving that this is the case. This is already difficult in the toric case where $V = \mathbb{C}^n$ and C is given as the closure of the image of a monomial map $f : \mathbb{C}^m \to \mathbb{C}^n$—all examples above are of this type. Using tropical geometry we have proved the following lower bound.

Theorem 1.5 ([8])**.** *Suppose that $f = (x^\alpha)_{\alpha \in A}$, where A is some subset of $\mathbb{N}^m$ of cardinality n. Assume that A lies on an affine hyperplane, so that $C := \overline{\operatorname{im} f}$ is indeed a cone. For any k-tuple $l = (l_1, \ldots, l_k)$ of affine-linear forms on $\mathbb{R}^m$ let $C_i(l)$ denote the subset of A where l_i is strictly smaller than all other $l_j, j \neq i$. Then*

$$\dim kC \geq \sum_{i=1}^{k} (1 + \dim \operatorname{Aff}_{\mathbb{R}} C_i),$$

where $\operatorname{Aff}_{\mathbb{R}} C_i$ is the affine span of C_i in $\mathbb{R}^m$.

To find good lower bounds with this theorem, one has to maximise the sum on the right-hand side over all k-tuples l, or, equivalently, over all regular subdivisions of $\mathbb{R}^m$ into k parts. In general this optimisation problem is not easy. Nevertheless, Baur and I have determined the secant dimensions of many embedded varieties in this manner.

Theorem 1.6 ([3]; see also [6])**.** *The secant varieties of $(\mathbb{P}^1)^i$ for $i = 1, 2, 3$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^2$, in all equivariant embeddings, are as expected, with an explicit list of exceptions.*

In her Master's thesis [4], Brannetti has reproved the Alexander-Hirschowitz theorem for $S^d(\mathbb{C}^4)$, for all d, with the method of [8]. These result lead to the following intriguing question.

Question 1.7. *Is the lower bound of Theorem 1.5, optimised over all k-tuples l, always the exact dimension of kC? I know of no counter-examples.*

Apart from these toric examples, we have also applied this approach to other *minimal orbits* X. Our results include a parameterisation of the cone C over X that when tropicalised hits a full-dimensional subset of the tropicalisation of C. For the smallest interesting case, where X is the collection of all incident point-line pairs in $\mathbb{P}^2$, we computed all secant dimensions of X in all SL_3-equivariant embeddings into projective spaces [3]. Related approaches to secant varieties, which also study their degrees and equations, are [7, 13].

2. Tropical algebraic groups

With Tyrrell McAllister I have initiated the study of tropicalising algebraic groups. There are many issues here: what coordinates to use? Does one expect a tropical multiplication on the result? I report some preliminary observations.

Proposition 2.1. *The tropicalisation of* SL_n*, with respect to matrix entries, is a monoid with respect to tropical matrix multiplication.*

Also, using Egerváry's theorem on minimal-weight matchings one can describe the maximal cones of this tropicalisation. For a not-so-easy example consider the orthogonal group $\mathrm{O}_n = \{g \mid gg^T = 1\}$. The choice for this non-split form is perhaps justified by the following beautiful observation.

Proposition 2.2. *The tropicalisation of* O_n *contains the matrices* $(d_{ij})_{ij}$ *satisfying* $d_{ii} = 0, d_{ij} = d_{ji}$*, and* $d_{ij} + d_{jk} \geq d_{ik}$*, as well as the closure of this set of matrices under tropical multiplication.*

(Note that these metric matrices form a cone of dimension $\binom{n}{2} = \dim \mathrm{O}_n$.) This is already rather interesting: combinatorially it is not clear why that closure should still have dimension $\binom{n}{2}$ (and not larger). This ends my preliminary account of tropical geometry of algebraic groups.

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