THE HILBERT NULL-CONE ON TUPLES OF MATRICES AND BILINEAR FORMS

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Abstract

We describe the null-cone of the representation of G on M^p, where either $G = \mathrm{SL}(W) \times \mathrm{SL}(V)$ and $M = \mathrm{Hom}(V, W)$ (linear maps), or $G = \mathrm{SL(V)}$ and M is one of the representations $S^2(V^*)$ (symmetric bilinear forms), $\Lambda^2(V^*)$ (skew bilinear forms), or $V^* \otimes V^*$ (arbitrary bilinear forms). Here V and W are vector spaces over an algebraically closed field K of characteristic zero and M^p is the direct sum of p of copies of M.

More specifically, we explicitly determine the irreducible components of the null-cone on M^p. Results of Kraft and Wallach predict that their number stabilises at a certain value of p, and we determine this value. We also answer the question of when the null-cone in M^p is defined by the polarisations of the invariants on M; typically, this is only the case if either $\dim V$ or p is small. A fundamental tool in our proofs is the Hilbert-Mumford criterion for nilpotency.

1. Introduction

For a group G and a finite-dimensional G-module M over an algebraically closed field K, we denote by $K[M]^G$ the algebra of G-invariant polynomials on M. An element $m \in M$ is called *nilpotent* if it cannot be distinguished from 0 by $K[M]^G$, or, in other words, if all G-invariant polynomials on M without constant term vanish on m. The nilpotent elements in M form a (Zariski-)closed cone in M, called the *null-cone* in M (G being understood) and denoted $\mathcal{N}(M) = \mathcal{N}_G(M)$; it is a central object of study in representation theory. In this paper we will describe the *irreducible components* of the null-cone in some concrete representations.

We will, in fact, be studying the null-cone in a direct sum M^p of p copies of M, regarded as a G-module with the diagonal action. We recall some relations between the invariants and the null-cone of M^q and those of M^p, where p and q are natural numbers. It is convenient, for this purpose, to identify M^p with $K^p \otimes M$ where G acts trivially on the first factor, and also, given a linear map $\pi : K^p \to K^q$, to use the same letter π for the G-homomorphism $M^p \to M^q$ determined by $\pi(x \otimes m) = \pi(x) \otimes m,\ x \in K^p, m \in M$.

First, from an invariant $f \in K[M^q]^G$ we can construct G-invariants on M^p as follows: for any linear map $\pi : K^p \to K^q$ the function $f \circ \pi$ is an invariant on M^p. The functions obtained in this way as π varies are usually called *polarisations* of f if $q \leq p$ and *restitutions* of f if $q \geq p$. Using this construction, due to Weyl [15], it is easy to see that any linear map $\pi : K^p \to K^q$ maps $\mathcal{N}(M^p)$ into $\mathcal{N}(M^q)$: indeed, an element v of the former null-cone cannot be distinguished from 0 by any

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G-invariants on M^p, let alone by those of the form $f \circ \pi$ with $f \in K[M^q]^G$; hence $\pi(v) \in \mathcal{N}(M^q)$. Using this observation, we can prove that the number $c(M^p)$ of irreducible components of the $\mathcal{N}(M^p)$ behaves as follows.

Proposition 1.1. *If $p \geq q$, then $c(M^p) \geq c(M^q)$. If in addition $q \geq \dim M$, then $c(M^p) = c(M^q)$ and the polarisations to M^p of the invariants on M^q without constant term define the null-cone set-theoretically.*

Proof. Fix any surjective linear map $\pi : K^p \to K^q$; we claim that it maps $\mathcal{N}(M^p)$ surjectively onto $\mathcal{N}(M^q)$. Indeed, if $\sigma : K^q \to K^p$ is a right inverse of π, then any $v \in \mathcal{N}(M^q)$ is the image under π of $\sigma v \in \mathcal{N}(M^p)$. This shows the first statement. For the second statement it suffices to prove that the map

$$\phi : \mathrm{Hom}(K^q, K^p) \times \mathcal{N}(M^q) \to \mathcal{N}(M^p),\ (\sigma, v) \mapsto \sigma v$$

is surjective for $q \geq \dim M$, because the right-hand side has precisely $c(M^q)$ irreducible components. To prove surjectivity of ϕ, let $v = (m_1, \ldots, m_p) \in \mathcal{N}(M^p)$. As $q \geq \dim M$, we can find a $w \in M^q$ whose components span the K-subspace $\langle m_1, \ldots, m_p\rangle_K$ in M. It follows that there exist linear maps $\pi : K^p \to K^q$ and $\sigma : K^q \to K^p$ such that $\pi v = w$ and $\sigma w = v$. We conclude that $w = \pi v$ lies in $\mathcal{N}(M^q)$ and $v = \phi(\sigma, w)$. The last statement is proved by a similar argument: suppose that all polarisations $f \circ \pi$ with $\pi \in \mathrm{Hom}(K^p, K^q)$ and $f \in K[M^q]^G$ without constant term vanish on $v \in M^p$, and let $h \in K[M^p]^G$ be without constant term. We can choose π and σ with $\sigma\pi v = v$ as before, and we find that $h(v) = ((h \circ \sigma) \circ \pi)v = 0$, because $(h \circ \sigma) \circ \pi$ is a polarisation of the G-invariant $h \circ \sigma$ on M^q. □

Remark 1.2. In characteristic zero the last statement of Proposition 1.1 also follows from from Weyl's stronger result that the invariant ring on M^p is generated by the polarisations of invariants on M^q for $q \geq \dim V$ [15]. Weyl's theorem no longer holds in positive characteristic, though a weaker statement is still true [9]. However, an analogue of Weyl's theorem, for *separating* invariants, is true in arbitrary characteristic [4]—and, again, implies the last statement of Proposition 1.1.

Proposition 1.1 shows that $c(M^p)$ is an ascending function of p that stabilises at some finite $p \leq \dim M$. This phenomenon was first observed by Kraft and Wallach in the case of reductive group representations [11], to which we turn our attention now. Suppose that G is a connected, reductive affine algebraic group over K and M is a rational finite-dimensional G-module. One of the most important results on the null-cone in this setting is the *Hilbert-Mumford criterion* [12, 13] for nilpotency: $v \in M$ lies $\mathcal{N}(M)$ if and only if there exists a one-parameter subgroup $\lambda : K^* \to G$ such that $\lim_{t\to 0} \lambda(t)v = 0$; we then say that λ *annihilates* v. In this setting much more can be said about the irreducible components of the null-cone in M^p: one verifies that for every one-parameter subgroup λ, the set

$$G \cdot \{v \in M^p \mid \lim_{t\to 0} \lambda(t)v = 0\} \tag{1}$$

is a closed G-stable irreducible subset of $\mathcal{N}(M^p)$, and that a finite number of them cover $\mathcal{N}(M^p)$. Moreover, for p sufficiently large, there are only the 'obvious' inclusions among these sets [11] and this observations gives rise to a combinatorial algorithm for counting the irreducible components of $\mathcal{N}(M^p), p >> 0$ [3]. However, for smaller values of p, there are usually many more inclusions, and our goal in this paper is to determine the exact 'stabilising' value of $c(M^p)$ for the pairs (G, M) in the abstract.

We note that the notion of 'optimal' one-parameter subgroups for elements of the null-cone gives yet a finer description of the geometry of $\mathcal{N}(M)$ [7, 13]—but this notion is not needed here.

Summarising, we will settle the following two fundamental problems for the pairs (G, M) of the abstract: first, we describe the irreducible components of $\mathcal{N}(M^p)$ and determine at which value of p their number stabilises; and second, we determine when $\mathcal{N}(M^p)$ is defined by the polarisations of the invariants on M. The remainder of this paper has the following transparent organisation: Sections 2, 3, 4, and 5 deal with tuples of linear maps, symmetric bilinear forms, skew bilinear forms, and arbitrary bilinear forms, respectively. In the rest of the text we assume that K has characteristic 0; this allows for the use of some 'differential' arguments in the case of linear maps, while avoiding problems in small characteristics in the case of bilinear forms. However, most of what is proved here remains valid in arbitrary characteristic.

2. Nilpotent tuples of linear maps

For an m-dimensional vector space V and an n-dimensional vector space W, both over our fixed algebraically closed field K of characteristic 0, the group $G = \mathrm{SL}(W) \times \mathrm{SL}(V)$ acts on the space $M = \mathrm{Hom}(V, W)$ of linear maps by $(g, h)A := gAh^{-1}$. By duality we may assume that $0 < m \leq n$, and we let $q := \lceil \frac{n}{m} \rceil$ be the smallest integer $\geq n/m$. Then $\mathcal{N}(M^p)$ is as follows.

Theorem 2.1. *The null-cone of* $\mathrm{SL}(W) \times \mathrm{SL}(V)$ *in* $M^p = \mathrm{Hom}(V, W)^p$ *consists of all p-tuples* $(A_1, \dots, A_p)$ *of linear maps for which there exist subspaces V' of V and W' of W such that* $n \cdot \dim V' > m \cdot \dim W'$ *and* $A_i V' \subseteq W'$ *for all i.*

The p-tuples for which V' can be chosen of a fixed dimension $k \in \{1, \dots, m\}$ *form a closed irreducible subset of* $\mathcal{N}(M^p)$*, denoted* $C_k^{(p)}$*. For $p < q$ the sets* $C_k^{(p)}$ *are all equal to M^p, and for $p > q$ they are precisely the distinct irreducible components of* $\mathcal{N}(M^p)$*. For $p = q$ there are still inclusions among the* $C_k^{(q)}$*, unless $m = 1$—in which case* $C_1^{(q)} = C_1^{(n)} = \mathcal{N}(M^n)$ *is the irreducible null-cone consisting of singular* $n \times n$*-matrices—or* $n = (q-1)m + 1$ *with* $q \geq 3$*, in which case the* $C_k^{(q)}$ *are already the distinct components of the null-cone.*

Somewhat prematurely, we will from now on call a pair V', W' as in the theorem a *witness* for the nilpotency of $(A_1, \dots, A_p)$. In the proof that follows we use a theorem from elementary optimisation theory, the *max-flow-min-cut theorem*, which states that the maximal size of a flow from a source s to a sink t in a network equals the minimal capacity of a cut disconnecting s from t; see [2, Chapter 3, Theorem 1] for details.

Proof of Theorem 2.1, part one. Suppose that $A = (A_1, \dots, A_p)$ lies in the null-cone and let $(\mu, \lambda) : K^* \to \mathrm{SL}(V) \times \mathrm{SL}(W)$ be a one-parameter subgroup annihilating A. Let $v_1, \dots, v_m$ be a basis of V with $\lambda(t)v_j = t^{a_j} v_j$, where $a_j \in \mathbb{Z}$, let $w_1, \dots, w_n$ be a basis of W with $\mu(t)w_i = t^{b_i} w_i$, where $b_i \in \mathbb{Z}$, and note that $\det \lambda(t) = \det \mu(t) = 1$ implies $\sum_j a_j = \sum_i b_i = 0$.

Now construct a directed graph Γ with arrows of capacity n from a source s to m vertices $1, \dots, m$, arrows of capacity m from n vertices $\hat{1}, \dots, \hat{n}$ to a sink t, and an

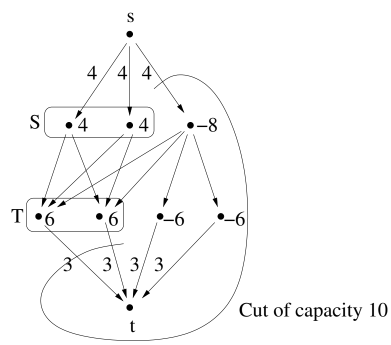

Figure 1. The graph Γ with a cut.

arrow—for convenience, of infinite capacity—from j to $\hat{i}$ if and only if $b_i - a_j > 0$. See Figure 1 for an example with $m = 4$ and $n = 6$. From

$$\lim_{t\to 0} \mu(t) A_k \lambda(t)^{-1} v_j = \lim_{t \to 0} \mu(t) A_k t^{-a_j} v_j = 0$$

it is clear that each A_k maps v_j into the space spanned by the w_i with $j \to \hat{i}$ in Γ. We claim that the maximal flow from s to t in Γ is strictly smaller than the obvious upper bound mn. Indeed, suppose that this upper bound were attained by a flow in which $c_{j,i}$ is the flow from j to $\hat{i}$. Then $\sum_i c_{j,i} = n$ for all j and $\sum_j c_{j,i} = m$ for all i, so that

$$0 = m \sum_i b_i - n \sum_j a_j = \sum_{j,i} c_{j,i}(b_i - a_j);$$

but $c_{j,i} = 0$ whenever $b_i - a_j \leq 0$, so that the right-hand side is strictly positive, a contradiction. Now the max-flow-min-cut theorem assures the existence of a cut of capacity strictly smaller than mn and in particular not containing edges of infinite capacity. Let $T \subseteq \{\hat{1}, \ldots, \hat{n}\}$ be the set of vertices cut off from t, and let $S \subseteq \{1, \ldots, m\}$ be the set of vertices *not* cut off from s. By definition of a cut, no vertex j of S is connected to any vertex $\hat{i}$ outside of T, so that $V' := \langle v_j \mid j \in S \rangle_K$ is mapped by every A_k into $W' := \langle w_i \mid \hat{i} \in T \rangle_K$. Finally, the capacity of the cut is equal to

$$m|T| + n(m - |S|) \text{ and by assumption } < mn,$$

so that $m \dim W' < n \dim V'$ as required.

Conversely, suppose that V', W' is a witness for the nilpotency of A, set $(k, l) := (\dim V', \dim W')$, and choose complements V'' and W'' of V' and W', respectively. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ having weights $a_1 := n(m - k)$ on V' and $a_2 := -nk$ on V''; note that $ka_1 + (n - k)a_2 = 0$. Similarly, let μ be the one-parameter subgroup of $\mathrm{SL}(W)$ having weights $b_1 := m(n - l)$ on W' and $b_2 := -ml$ on W''. From the inequalities

$$b_1 - a_1 > 0,\ b_1 - b_2 > 0,\ b_2 - a_1 \leq 0, \text{ and } b_2 - a_2 > 0$$

we infer that (μ, λ) annihilates any linear map sending V' into W', so that $A \in \mathcal{N}(M^p)$. This proves the first statement of the theorem. □

The sets $C_k^{(p)}$ from Theorem 2.1 are closed and irreducible by a general argument: they are of the form (1). Hence to prove the theorem we need only determine for what values of p there are inclusions among the $C_k^{(p)}$. For this we need some auxiliary notation and results, which are of independent interest and which also give a formula for the dimensions of the irreducible components of $\mathcal{N}(M^p)$. We write $M_{a,b}$ for the space of $a \times b$-matrices with entries in K.

Definition 2.2. Let a, b, c, d, and p be non-negative integers and let

$$X_i \in M_{c,a} \text{ and } Y_i \in M_{b,d} \text{ for } i = 1, \ldots, p.$$

Define the *cut-and-paste map* $\mathrm{CP} = \mathrm{CP}_{(X_i,Y_i)_i} : M_{a,b} \to M_{c,d}$ by

$$\mathrm{CP}\, A = \sum_{i=1}^{p} X_i A Y_i.$$

Now the rank of the linear map CP is clearly a lower semi-continuous function of the p-tuple $(X_i, Y_i)_i$, and we let $\mathrm{cp}^{(p)}(a, b, c, d)$, the *cut-and-paste rank*, be the maximal possible rank of CP, i.e., the rank for a generic p-tuple $(X_i, Y_i)_i$.

Remark 2.3. The following properties of the cut-and-paste rank are easy to check:

$$\mathrm{cp}^{(p)}(c, d, a, b) = \mathrm{cp}^{(p)}(a, b, c, d) = \mathrm{cp}^{(p)}(b, a, d, c).$$

Indeed, the second equality comes from the fact that, upon composition with transposition on both sides, the cut-and-paste map $\mathrm{CP}_{(X_i,Y_i)_i} : M_{a,b} \to M_{c,d}$ yields $\mathrm{CP}_{(Y_i^t,X_i^t)_i} : M_{b,a} \to M_{d,c}$; and the first equality reflects the fact that the transpose of $\mathrm{CP}_{(X_i,Y_i)_i}$ can be identified, via the trace form, with $\mathrm{CP}_{(X_i^t,Y_i^t)_i} : M_{c,d} \to M_{a,b}$. Moreover, if $a \leq c$ and $b \leq d$ then $\mathrm{cp}^{(p)}(a, b, c, d) = ab$ for all $p \geq 1$. Thus we reduce the computation of the cut-and-paste-rank to the case where $ab \leq cd, a \geq c$, and $b \leq d$. Then each of the maps $A \mapsto X_i A Y_i$ generically has rank bc, so that

$$\mathrm{cp}^{(p)}(a, b, c, d) \leq \min\{ab, pbc\}$$

Moreover, for $p \leq a/c$ it is easy to see that $\mathrm{cp}^{(p)}(a, b, c, d)$ is in fact equal to pbc: by using suitable X_i and Y_i, one can 'cut' p non-overlapping $c \times b$-blocks from an $a \times b$-matrix, and 'paste' them in a non-overlapping way into a $c \times d$-matrix. The same argument shows that for p sufficiently large $\mathrm{cp}^{(p)}(a, b, c, d)$ equals ab; this is the case, for example, as soon as one can cut an $a \times b$-matrix into p non-overlapping rectangular blocks that fit without overlap into a $c \times d$-matrix. One might think that the inequality for the cut-and-paste-rank given above is always an equality, but this is not true: for $(a, b, c, d) = (5, 4, 3, 7)$, for instance, we find cut-and-paste-ranks $12, 19, 20$ for $p = 1, 2, 3$, respectively. In short, we have no closed formula for cp and it would be interesting—but too much of a digression at this point in the paper—to find such a formula. In small concrete cases, however, the cut-and-paste rank can be computed easily; see below for some examples

Proposition 2.4. *Let k, l, m, n, p be integers satisfying $0 < k \leq m$, $0 \leq l < n$, and $p \geq 0$. Then*

$$Q := \{(A_1, \ldots, A_p) \in M_{n,m}^p \mid \exists U \subseteq K^m : \dim U = k \text{ and } \dim(\sum_{i=1}^{p} A_i U) \leq l\}.$$

is an irreducible variety, and a sufficient condition for Q to be strictly smaller than $M_{n,m}^p$ is

$$p > \frac{l}{k} + \frac{m-k}{n-l}.$$

Moreover, $\dim Q$ *equals*

$$\begin{cases} pmn \text{ if } pk \leq l, \text{ and} \\ pmn - (pk-l)(n-l) + \mathrm{cp}^{(p)}(m-k, k, \min\{p(m-k), n-l\}, pk-l), \text{ otherwise.} \end{cases}$$

Proof. The set Q is an irreducible variety because it is of the form (1), that is, the result a vector space stable under a Borel subgroup of $G = \mathrm{SL}_n \times \mathrm{SL}_m$ being 'smeared' around by G. For $pk \leq l$ the proposition is evident: any p-tuple maps any k-space into an l-space. Suppose therefore that $pk \geq l$. In the diagram

$$\begin{array}{ccc} M_{n,m}^p \times (M_{m,k})_{\mathrm{reg}} & \xrightarrow{\mu} & M_{n,pk} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \\ M_{n,m}^p & & \end{array}$$

μ maps $(A_1, \ldots, A_p, B)$ to $(A_1 B| \ldots |A_p B)$, $\tilde{\pi}$ is the projection, and $(M_{n,k})_{\mathrm{reg}}$ is the set of rank k matrices. Hence $Q = \tilde{\pi}(\mu^{-1}(X_l))$, where X_l is the variety of matrices in $M_{n,pk}^p$ having rank at most l. We will first compute the dimension of $Z := \mu^{-1}(X_l)$ and then the dimension of a generic fibre of $\pi := \tilde{\pi}|_Z : Z \to Q$; the difference between these numbers is the dimension of Q.

First, μ is surjective and all its fibres have the same dimension $km + pn(m-k)$. Indeed, for $(A_1, \ldots, A_p, B)$ to lie in the fibre over $(C_1, \ldots, C_p)$ we may choose $B \in (M_{m,k})_{\mathrm{reg}}$ arbitrarily, and then each A_i is determined on the k-dimensional image of B, but can still be freely prescribed on an $(n-k)$-dimensional complement. As X_l has dimension $nl + pkl - l^2$ [6], Z has dimension $km + pn(m-k) + nl + pkl - l^2$. Now GL_k acts faithfully on the fibres of π by $g((A_i)_i, B) := ((A_i)_i, Bg^{-1})$, so that

$$\dim Q = \dim \pi(Z) \leq \dim Z - k^2 = pnm - (pk(n-l) - k(m-k) - l(n-l)).$$

This implies the first statement of the proposition.

For the dimension of Q we compute the dimension of a generic fibre $\pi^{-1}\pi(z)$ by computing $T_z \pi^{-1}\pi(z)$, as follows. First, we show that Z is irreducible and determine $T_z Z$ for generic $z \in Z$. Observe for this that the group GL_m acts on the fibres of μ by $g((A_i)_i, B) := ((A_i g^{-1})_i, gB)$. Now the map

$$\phi : \mathrm{GL}_m \times M_{n,pk} \times M_{n,m-k}^p \to M_{n,k}^p \times M_{m,k},$$
$$(g, (C_1| \ldots |C_p), (E_i)_i) \mapsto g((C_i|E_i)_i, \left(\frac{I_k}{0_{m-k,k}}\right))$$

maps $\mathrm{GL}_m \times X_l \times M_{n,m-k}^p$ surjectively onto Z, so Z is irreducible as claimed. Furthermore, the map

$$s : M_{n,m-k}^p \to M_{n,k}^p \times M_{m,k}, \; x \mapsto \phi(1, x, (0)_i)$$

is a right inverse of μ, so by the chain rule $d_z\mu$ maps $M_{n,m}^p \times M_{m,k}$ surjectively onto $T_{\mu(z)} X_l$ for all $z \in M_{n,m}^p \times M_{m,k}$. In particular, if z lies in Z and $\mu(z)$ has rank exactly l so that it is a smooth point of X_l, then we have

$$T_z Z = (d_z \mu)^{-1} T_{\mu(z)} X_l. \tag{2}$$

Now recall that if $\mu(z)$ has rank l, then

$$T_{\mu(z)} X_l = \{N \in M_{n,pk} \mid N \ker \mu(z) \subseteq \operatorname{im} \mu(z)\}; \tag{3}$$

see [6, Example 14.16]. This will enable us to interpret the right-hand side in (2). On the other hand, because $\operatorname{char} K = 0$, we have

$$T_z\pi^{-1}\pi(z) = \ker(d_z\pi : T_zZ \to T_{\pi(z)}Q) \tag{4}$$

for generic $z \in Z$. Now let $z = ((A_i)_i, B) \in Z$ be generic. In particular, we require (2) and (4), and what further open conditions on z are needed will become clear along the way. By the action of GL_m above we may assume that B is of the form

$$B = \begin{bmatrix} I_k \\ 0_{m-k,k} \end{bmatrix},$$

and we split each $A_i = (A_{i,1}|A_{i,2})$, accordingly. By genericity of the A_i the matrix $\mu(z) = (A_{1,1}|\dots|A_{p,1})$ has rank l, and by (2), (3), and (4) we find that $T_z\pi^{-1}(\pi(z))$ is isomorphic to the space of all $m \times k$-matrices

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

such that

$$(A_{1,1}D_1 + A_{1,2}D_2|\dots|A_{p,1}D_1 + A_{p,2}D_2) \ker \mu(z) \subseteq \operatorname{im} \mu(z).$$

This is clearly the case for $D_2 = 0$ (this reflects the GL_k-action used earlier), hence to determine what other D have this property we may assume that $D_1 = 0$. The kernel of $\mu(z)$ has dimension $pk-l$, so we can choose p matrices $Y_1, \dots, Y_p \in M_{k,pk-l}$ such that the columns of the matrix

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}$$

form a basis of the kernel of $\mu(z)$. Again by genericity—the $A_{i,2}$ are 'independent' of the $A_{i,1}$—the pre-image of $\operatorname{im} \mu(z)$ under $(A_{1,2}|\dots|A_{p,2})$ has codimension $c := \min\{p(m-k), n-l\}$ in $K^{p(m-k)}$, and we may choose matrices $X_1, \dots, X_p \in M_{c,m-k}$ such that the rows of $(X_1|\dots|X_p)$ give linear equations for that inverse image. We now have

$$\begin{aligned} &\{D_2 \in M_{m-k,k} \mid (A_{1,2}D_2|\dots|A_{p,2}D_2) \ker(A_{1,1}|\dots|A_{p,1}) \subseteq \operatorname{im}(A_{1,1}|\dots|A_{p,1})\} \\ &= \{D_2 \in M_{m-k,k} \mid \sum_i X_i D_2 Y_i = 0\} \\ &= \ker(\mathrm{CP}_{(X_i,Y_i)_i} : M_{m-k,k} \to M_{c,pk-l}). \end{aligned}$$

Finally, because the X_i and Y_i are generic along with the A_i, the dimension of this space is $(m-k)k - \mathrm{cp}^{(p)}(m-k, k, c, pk-l)$. The dimension of the fibre $\pi^{-1}(\pi(z))$ is therefore k^2 plus this number, and we find

$$\begin{aligned} \dim \pi(Z) = &\dim Z - \dim \pi^{-1}\pi(z) \\ =&km + pn(m-k) + nl + pkl - l^2 \\ &- k^2 - ((m-k)k - \mathrm{cp}^{(p)}(m-k, k, \min\{p(m-k), n-l\}, pk-l)) \\ =&pmn - (pk-l)(n-l) + \mathrm{cp}^{(p)}(m-k, k, \min\{p(m-k), n-l\}, pk-l), \end{aligned}$$

as claimed. □

Remark 2.5. The difference $\dim \pi^{-1}(\pi(z)) - k^2$, expressed above as the nullity of a certain cut-and-paste map, is the dimension of the variety of k-dimensional subspaces U for which $\sum_i A_i U$ is at most l-dimensional.

Example 2.6. Proposition 2.4 is particularly useful to prove the existence of tuples of matrices not mapping any subspace of dimension k into a subspace of dimension l. Consider the following two questions.

(1) Do all triples (A_1, A_2, A_3) of 8×5-matrices map some 4-dimensional subspace into some 7-dimensional subspace? Set $(m, n, k, l, p) = (5, 8, 4, 7, 3)$ and compute

$$\frac{l}{k} + \frac{m-k}{n-l} = \frac{7}{4} + \frac{1}{1} < 3 = p,$$

hence by the proposition the answer is no: there exist triples (A_1, A_2, A_3) such that for all U of dimension 4 we have $\sum A_i U = K^8$. This may not come as a surprise; however, it is not entirely obvious how to construct such a 'generic' triple. For instance, we cannot choose them such that each A_i is monomial in the sense that it maps every standard basis vector of K^5 to some multiple of a standard basis vector of K^8: if this is the case, then the inequality $8 \cdot 2 > 5 \cdot 3$ implies that there is a basis vector e_i of K^8 which is 'hit only once' by some A_p applied to some e_k. But then $U = \bigoplus_{l \neq k} K e_l$ is mapped into $\bigoplus_{j \neq i} K e_j$.

(2) Do all triples of 5×5-matrices map some 2-dimensional space into some 3-dimensional space? Set $(m, n, k, l, p) = (5, 5, 2, 3, 3)$ in the proposition. Now we find

$$\frac{l}{k} + \frac{m-k}{n-l} = \frac{3}{2} + \frac{3}{2} = 3 = p,$$

so we need a more detailed analysis. The cut-and-paste rank in the proposition is

$$\mathrm{cp}^{(3)}(3, 2, 2, 3),$$

which is $3 \cdot 2 = 6$ as one can cut a 3×2-matrix into $p = 3$ rectangular pieces that can be put together without overlap to make up a 2×3-matrix. It follows that the dimension in the proposition is in fact pmn, i.e., that indeed, every triple of 5×5-matrices maps some 2-dimensional space into some 3-dimensional space. To prove this is a nice exercise for students in linear algebra. (It is also true in positive characteristic.)

Proof of Theorem 2.1, part two. It is clear that if $p < q := \lceil \frac{n}{m} \rceil$, then for *any* subspace V' of V we have $\dim(\sum_{i=1}^p A_i V') \leq p \dim V' < \frac{n}{m} \dim V'$, so that all $C_k^{(p)}$ are equal to $M^p = \mathrm{Hom}(V, W)^p$. In other words: there are no invariants on M^p for $p < q$.

Next suppose that $p \geq q + 1$; then we have to show that there are no inclusions among the $C_k^{(p)}$. For every $k \in \{1, \ldots, m\}$ let $l_k := \lceil k \frac{n}{m} \rceil - 1$ denote the maximal $l \in \{0, \ldots, n-1\}$ with $\frac{l}{k} < \frac{n}{m}$. One readily verifies that $1 \leq l_{k+1} - l_k \leq q$ for all $k \in \{1, \ldots, m-1\}$ (the first inequality follows from our standing assumption $n \geq m$). Fix $k \in \{1, \ldots, m\}$ and set $l := l_k$, so that every p-tuple in $C_k^{(p)}$ maps some k-space into an l-space. We will construct a p-tuple $(A_1, \ldots, A_p)$ lying in $C_k^{(p)}$

and not in any $C_{k'}^{(p)}$ with $k' \neq k$, as follows: Every A_i will have the block form

$$A_i = \begin{bmatrix} A_i' & \\ & A_i'' \end{bmatrix},$$

where A_i' is an $l \times k$-matrix and A_i'' is an $(n-l) \times (m-k)$-matrix. The A_i' will have the property that

$$\dim(\sum_i A_i' U') \geq \frac{n}{m} \dim U' \text{ for all } \textit{proper} \text{ subspaces } U' \subsetneq K^k,$$

and $\dim(\sum_i A_i K^k) = l$. The A_i'' will have the property that

$$(l + \dim(\sum_i A_i'' U'')) \geq \frac{n}{m}(k + \dim U'') \text{ for all } \textit{non-zero} \text{ subspaces } 0 \neq U'' \subseteq K^{m-k};$$

note that, by the choice of l, this latter inequality is then still valid when l on the left and k on the right are replaced by 0.

Suppose that such A_i' and A_i'' exist and let the A_i be the block matrices above. Let U be a subspace of K^m unequal to K^k. Let U' be the intersection of U with K^k and let U'' be the projection of U on K^{m-k} along K^k. Then $\dim U = \dim U' + \dim U''$ and one readily sees that

$$\dim(\sum_i A_i U) \geq \dim(\sum_i A_i' U') + \dim(\sum_i A_i'' U''). \tag{5}$$

Now there are two possibilities: either $U' \neq K^k$, or $U' = K^k$ but $U'' \neq 0$. In the first case one finds that the right-hand side is at least

$$\frac{n}{m} \dim U' + \frac{n}{m} \dim U'' = \frac{n}{m} \dim U.$$

If, on the other hand, $U' = K^k$ but $U'' \neq 0$, then we find that the right-hand side in (5) is at least

$$l + \dim(\sum_i A_i'' U'') \geq \frac{n}{m}(k + \dim U'') = \frac{n}{m} \dim U.$$

In other words, with A_i, A_i', A_i'' as above the pair (K^k, K^l) is the *only* witness for the nilpotency of $(A_1, \ldots, A_p)$, and *a fortiori* this p-tuple lies in a unique $C_k^{(p)}$.

To find the A_i' we show that for all $k' \in \{1, \ldots, k-1\}$ and $l' \in \{0, \ldots, l-1\}$ with $\frac{l'}{k'} < \frac{n}{m}$ the dimension of the set of p-tuples $(A_1', \ldots, A_p') \in M_{l,k}$ that map a k'-space into an l'-space is smaller than plk. To this end we want to apply the sufficient condition of Proposition 2.4 with m, n, k, l replaced by k, l, k', l', respectively. Compute therefore

$$\frac{l'}{k'} + \frac{k - k'}{l - l'} < \frac{n}{m} + 1 \leq q + 1 \leq p,$$

where for the second term we used $l' \leq l_{k'}$ and the strict increasingness of the l_k. This shows the existence of $A_1', \ldots, A_p'$ as required.

Similarly, to find the A_i'' we show that for all $k' \in \{k+1, \ldots, m\}$ and $l' \in \{l, \ldots, n-1\}$ with $\frac{l'}{k'} < \frac{n}{m}$ there exists there exists a p-tuple $(A_1'', \ldots, A_p'') \in M_{m-k,n-l}$ that does not map any $(k'-k)$-dimensional space into an $l'-l$-dimensional space. Again, we apply the proposition, but now with m, n, k, l replaced by $m-k, n-l, k'-k, l'-l$, respectively. Consider therefore the expression

$$\frac{l' - l}{k' - k} + \frac{m - k'}{n - l'}$$

As $l' \leq l_{k'}$ and $l = l_k$ the first term is at most q. On the other hand, as $l' < \frac{n}{m}k'$, the denominator of the second term satisfies

$$n - l' > n - \frac{n}{m}k' = \frac{n}{m}(m - k') \geq m - k',$$

hence the second term smaller than 1. We conclude that

$$p \geq q + 1 > \frac{l' - l}{k' - k} + \frac{m - k'}{n - l'},$$

hence by Proposition 2.4 there exists a p-tuple as required, and this concludes the case where $p > q$.

Finally, we assume that $p = q$. First suppose that there exists a $k \in \{1, \ldots, m-1\}$ with $l_{k+1} - l_k = q$. Then any q-tuple $(A_1, \ldots, A_q) \in C_k^{(q)}$ maps a k-space into an l_k-space, and adding one arbitrary dimension to that k-space yields a $(k+1)$-space mapped by all A_i into a space of dimension $l_k + q = l_{k+1}$. In other words, we have $C_k^{(q)} \subseteq C_{k+1}^{(q)}$, so that there are indeed inclusions among the $C_k^{(q)}$. Next suppose that no such k exists. Then we have

$$n - 1 = l_m \leq l_1 + (m-1)(q-1) = m(q-1) < m\frac{n}{m} = n,$$

so that $n = m(q-1) + 1$, where $q \geq 2$. In this case $l_k = (q-1)k$ for all k, and for $q > 2$ the inequalities

$$\frac{l_{k'}}{k'} + \frac{k - k'}{l_k - l_{k'}} = (q-1) + \frac{1}{q-1} < q, \ k' < k$$

and

$$\frac{l_{k'} - l_k}{k' - k} + \frac{m - k'}{n - l_{k'}} = (q-1) + \frac{m - k'}{(q-1)(m-k') + 1} < q, \ k' > k$$

readily imply that the construction of the A_i above still works to show that $C_k^{(q)}$ is not contained in any other $C_{k'}^{(q)}$. The last case to be considered is $q = 2$ and $n = m + 1$. Then $l_k = k$ for all k, and any pair of matrices mapping a k-space into a k-space also maps a $(k-1)$-space into a $(k-1)$-space, so that the null-cone on $q = 2$ copies is irreducible. □

As promised in the Introduction, we now investigate when the polarisations of invariants on one copy of $\mathrm{Hom}(V, W)$ define the null-cone on p copies. This question is interesting only in the case where there *are* non-trivial invariants on one copy—hence if $\dim V = \dim W$, in which case we may as well assume $V = W$. Then the invariant ring is generated by the determinant on $\mathrm{End}(V)$ (see, e.g., [10, Section I.3]).

Theorem 2.7. *The null-cone in* $\mathrm{End}(V)^p$ *is defined by the polarisations of* det *if and only if* $\dim V \leq 2$ *or* $p \leq 2$.

Proof of Theorem 2.7. The result for $p = 2$ follows from the Kronecker-Weierstrass theory of matrix pencils, see [5]; for completeness we include a short proof in our terminology. By Theorem 2.1 we have to show that if $A, B \in \mathrm{End}(V)$ satisfy $\det(sA + tB) = 0$ for all $s, t \in K$, then there exists a witness $V', W' \subseteq V$ for the nilpotentcy of (A, B). Indeed, regarding s, t as variables, $sA + tB$ has a non-zero vector $u(s,t)$ in $K[s,t] \otimes_K V$ in its kernel. But then any non-zero homogeneous component of $u(s,t)$, say of degree d, is also annihilated by $sA + tB$; hence we find

$u_0, \ldots, u_d \in V$ such that $(sA+tB)(s^d u_0 + s^{d-1}tu_1 + \ldots + t^d u_d) = 0$, where we may assume that $u_0 \neq 0$. Taking the of coefficients of $s^{d+1}, s^d t, \ldots, t^{d+1}$, we find

$$Au_0 = 0,\ Au_1 = -Bu_0,\ \ldots,\ Au_d = -Bu_{d-1},\ \text{and } Bu_d = 0.$$

But then every element of $KA + KB$ maps the space $V' := \sum_i Ku_i$ into the space $U' := \sum_i KAu_i$, which is strictly smaller because $Au_0 = 0$ while $u_0 \neq 0$.

The statement for $\dim V = 2$ is easy: in a linear space of matrices of rank ≤ 1 either all matrices have the same image, or all matrices have the same kernel. Now suppose that $m, n \geq 3$. To show that the null-cone in $\mathrm{End}(V)^m$ is then *not* defined by the polarisations of det, it suffices to construct a 3-dimensional singular subspace of $\mathrm{End}(V)$ for which there do not exist V', W' as above. The space

$$\left\{ \begin{bmatrix} 0 & a & b & & & & \\ -a & 0 & c & & & & \\ -b & -c & 0 & & & & \\ & & & a & & & \\ & & & & a & & \\ & & & & & \ddots & \\ & & & & & & a \end{bmatrix} \mid a, b, c \in K \right\} \text{(empty entries are always zero)},$$

is such a space, as one easily verifies. □

3. $\mathrm{SL}(V)$ on symmetric bilinear forms

The group $\mathrm{SL}(V)$ acts on bilinear forms as follows: if α is a bilinear form and $g \in \mathrm{SL}(V)$, then $(g\alpha)(v, w) = \alpha(g^{-1}v, g^{-1}w)$. It will be convenient to associate to every bilinear a linear map as follows: we fix, once and for all, a non-degenerate, *symmetric* bilinear form $(.,.)$ on V, and denote the *transpose* of $A \in \mathrm{End}(V)$ relative to this form by A^t. If α is a bilinear form on V, then we associate to α a linear map A by the requirement that $\alpha(x, y) = (x, Ay)$ for all $x, y \in V$. Then g acts on A by $g \cdot A := (g^{-1})^t A g^{-1}$. Note that the image of $\mathrm{SL}(V)$ in $\mathrm{GL}(\mathrm{End}(V))$ under this representation is contained in the image of $\mathrm{SL}(V) \times \mathrm{SL}(V)$ under the representation of Section 2.

As in Section 2 the invariants of $\mathrm{SL}(V)$ on $S^2(V^*)$ are generated by the determinant of (the linear map associated to) the form, and the null-cone on one copy is therefore the irreducible variety of singular forms.

Theorem 3.1. *For $p \geq 2$ and $n := \dim V$, the null-cone of $\mathrm{SL}(V)$ on $S^2(V^*)^p$ has $\lfloor \frac{n+1}{2} \rfloor$ irreducible components given by*

$$C_k^{(p)} := \{(\alpha_1, \ldots, \alpha_p) \mid \exists U \subseteq W \subseteq V : \dim U = k, \dim W = n - k + 1,\ \textit{and}$$
$$\alpha_i(U, W) = 0 \textit{ for all } i = 1, \ldots, p\},\ k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor.$$

Suppose that $(\alpha_1, \ldots, \alpha_p)$ lies in $C_k^{(p)}$, and that U and W are a witness of its nilpotency as in the theorem. A dimension argument shows that U must intersect the radical of each α_i non-trivially; in particular, if α_i has rank $n-1$, then its radical is contained in U, and W is precisely $U^{\perp_{\alpha_i}} := \{v \in V \mid \alpha_i(U, v) = 0\}$; we refer to this space as the *orthoplement* of U relative to α_i.

Suppose now that all α_i have rank $n-1$. Then a geometric interpretation of U, W as in the theorem is the following: $\mathbb{P}U$ is a linear subspace of $\mathbb{P}V$ common to all quadrics $Q_i = \{x \in \mathbb{P}V \mid \alpha_i(v, v) = 0\}$ and containing their radicals, and for each i, $\mathbb{P}W$ is the space tangent to Q_i at all of $\mathbb{P}U$. For example, if $n = 4$ and $p = 2$, then a pair (α_1, α_2) of rank 3 forms lies in C_1 if and only if α_1 and α_2 have the same radical (a projective point); if $(\alpha_1, \alpha_2) \notin C_1$, then the pair lies in C_2 if and only if the quadrics Q_1, Q_2 are tangent along the (projective) line through their radicals. This interpretation will yield a nice proof of the following theorem.

Theorem 3.2. *The null-cone on $S^2(V^*)^p$ is defined by the polarisations of* det *if and only if* $\dim(V) \leq 4$ *or* $p \leq 2$.

Remark 3.3. The description of the null-cone in Theorem 3.1 already appears in [14, Theorem 0.1(ii)]. However, Wall claims in Corollary 1 of *loc. cit.* that the null-cone on *any* number of copies is defined by the polarisations of det—which, as we will see below, is false for $n \geq 5$.

First, however, we proof Theorem 3.1. In contrast to our proof for tuples of matrices, we will give explicit pairs of symmetric forms representing the various components of the null-cone; for this the following lemma is useful.

Lemma 3.4. *Let m, n, k be non-negative integers and let $\pi_1, \ldots, \pi_p$ be partially defined strictly increasing functions $\{1, \ldots, m\} \to \{1, \ldots, n\}$, that is, every π_l is defined on a subset $\mathrm{dom}(\pi_l)$ of $\{1, \ldots, m\}$ and satisfies*

$$i < j \Rightarrow \pi_l(i) < \pi_l(j) \text{ whenever the right-hand side is defined.}$$

For $l = 1, \ldots, p$ let $A_l : K^m \to K^n$ be a linear map mapping e_i to a non-zero multiple of $e_{\pi_l(i)}$ if π_l is defined at i, and to zero otherwise. Let U be a subspace of K^m and set

$$\mathrm{gr}\, U := \{i \in \{1, \ldots, m\} \mid U \cap (e_i + \langle e_1, \ldots, e_{i-1}\rangle_K) \neq \emptyset\}.$$

Then

$$\dim \sum_l A_l U \geq \left| \bigcup_l \pi_l(\mathrm{gr}\, U \cap \mathrm{dom}\, \pi_l) \right|$$

We will call a p-tuple $(A_1, \ldots, A_p)$ of linear maps as in this lemma *standard.*

Proof. We have $|\mathrm{gr}(U)| = \dim U$, and defining $\mathrm{gr}\, W$ for subspaces W of K^n in a similar way the conditions on the A_i guarantee that

$$\mathrm{gr}(\sum_l A_l U) \supseteq \bigcup_l \pi_l(\mathrm{gr}\, U \cap \mathrm{dom}\, \pi_l),$$

whence the lemma follows immediately. □

Proof of Theorem 3.1. Suppose that $(\alpha_1, \ldots, \alpha_p)$ lies in the null-cone, and let A_i be the matrix associated to α_i. Then $(A_1, \ldots, A_p)$ lies in the null-cone of $\mathrm{SL}(V)$ acting on $\mathrm{End}(V)$ as indicated above and, *a fortiori*, in the null-cone of $\mathrm{SL}(V) \times \mathrm{SL}(V)$ on $\mathrm{End}(V)$ discussed in Section 2. Hence by Theorem 2.1 there exist subspaces U' and W' of V with $\dim W' = n - \dim U' + 1$ and such that every A_i maps U' into the orthoplement of W' relative to $(.,.)$. But then $\alpha_i(w, u) = (w, A_i u) = 0$ for all $u \in U'$ and $w \in W'$. Now set $U := U' \cap W'$ and $W := U' + W'$. Then clearly $U \subseteq W$, $\dim U + \dim W = \dim U' + \dim W' = n + 1$, and $\alpha_i(U, W) = 0$ for all i.

The $C_k^{(p)}$ are closed and irreducible as usual (see the Introduction), and so it only remains to check that there are no inclusions among them for $p \geq 2$. To this end, let $k \in \{1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$; we will construct a pair $(\alpha, \beta) \in C_k^{(2)}$ that does not lie in any $C^{(2)}_{k'}$ with $k \neq k'$. Take $V = K^n$ and $(x, y) := \sum_{i=1}^{n} x_i y_{n+1-i}$, so that transposition relative to this form corresponds to reflection of the matrix in the 'skew diagonal'. Now take the standard pair (A, B) of for which

$$
sA + tB = \left[\begin{array}{c|c|c}
\begin{matrix} s & t & & \\ & \ddots & \ddots & \\ & & s & t \end{matrix} & & \\ \hline
& \begin{matrix} t & & & \\ s & \ddots & & \\ & \ddots & \ddots & \\ & & s & t \end{matrix} & \\ \hline
& & \begin{matrix} t & & \\ s & \ddots & \\ & \ddots & t \\ & & s \end{matrix}
\end{array}\right],
$$

where the diagonal block sizes are, from top left to bottom right, $(k-1) \times k$, $(n-2k+1) \times (n-2k+1)$, and $k \times (k-1)$. Let α and β be the forms defined by A and B, respectively. Now if U and W are subspaces of K^n with $\dim U + \dim W = n+1$ and $\alpha(U, W) = \beta(U, W) = 0$, then one finds $\dim(AU + BU) < \dim U$. But by Lemma 3.4 the only pair subspaces of K^n having this property are $U = \langle e_1, \ldots, e_k \rangle_K$ and $W = \langle e_1, \ldots, e_k, \ldots, e_{n-k+1} \rangle_K$. This shows that (U, W) is the unique witness for the nilpotency of (α, β), and hence (α, β) does not lie in any other component $C^{(2)}_{k'}$. □

Proof of Theorem 3.2. On $p = 2$ copies the null-cone is defined by the polarisations of the determinant. This follows either from the Kronecker-Weierstrass theory of pencils of forms [5] or from a direct construction of U and W as in Theorem 3.1 for any two-dimensional space of singular forms.

Next we prove that for $n \leq 4$ the null-cone on *any* number p of copies is defined by the polarisations of det, or, in other words, that any space $\mathcal{A}$ of singular symmetric bilinear forms is spanned by a tuple $(\alpha_1, \ldots, \alpha_p)$ lying in some $C^{(p)}_k$; slightly inaccurately, we will then say that $\mathcal{A}$ *lies in* C_k. Note that we need only prove this for maximal spaces of singular forms; in particular, we may assume that $\mathcal{A}$ contains forms of rank $n-1$, because if it does not, we may add any rank 1 form to $\mathcal{A}$ without creating non-degenerate forms. In what follows we heavily use the fact that any 2-dimensional space of singular forms does already lie in some C_k.

For $n = 2$, the quadric of a rank 1 form is a point on the projective line $\mathbb{P}V$. As for any two non-zero forms in $\mathcal{A}$ this point coincides, it is the same for *all* forms in $\mathcal{A}$. Hence $\mathcal{A}$ lies in C_1.

For $n = 3$, the quadric of a rank 2 form α is the union of two lines in the projective plane $\mathbb{P}V$, whose intersection is the radical of α. If the radicals of any two forms in $\mathcal{A}$ of rank 2 coincide, then $\mathcal{A}$ lies in C_1; suppose, therefore, that there exist forms α_0, α_1 in $\mathcal{A}$ of rank 2 whose radicals are distinct. We have $(\alpha_0, \alpha_1) \in C_2$, so that their quadrics Q_0 and Q_1 have a line L in common (see Figure 2). Now a generic element $\beta \in \mathcal{A}$ has rank 2, does not have the same radical as α_0 or α_1, and its quadric Q_β is not the union of the non-common lines of Q_0 and Q_1. But Q_β

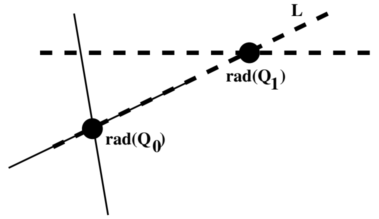

FIGURE 2. Proof of Theorem 3.2 for $n = 3$

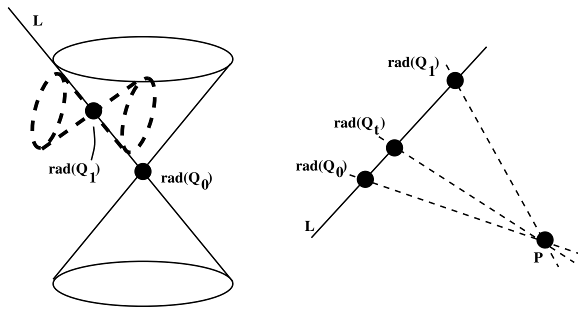

FIGURE 3. Proof of Theorem 3.2 for $n = 4$

must have lines in common with both Q_0 and Q_1, and therefore it contains L. But then L is isotropic relative to all forms in $\mathcal{A}$, and $\mathcal{A}$ lies in C_2.

For $n = 4$, suppose that there exist forms $\alpha_0, \alpha_1 \in \mathcal{A}$ of rank 3 whose radicals do not coincide (otherwise $\mathcal{A}$ lies in C_1). The corresponding quadrics $Q_0, Q_1 \subseteq \mathbb{P}V$ are tangent along the line L connecting their radicals (see Figure 3, left). For $t \in K$ set $\alpha_t := (1-t)\alpha_0 + t\alpha_1$ and

$$T := \{t \in K \mid \mathrm{rk}(\alpha_t) = 3\}.$$

For each $t \in T$, the quadric Q_t of α_t is tangent to Q_0 along L, and its radical lies on L; the set of all radicals thus obtained forms a dense set of L.

If all rank 3 forms in $\mathcal{A}$ have their radicals on L, then their quadrics are all tangent to Q_0 along L and $\mathcal{A}$ lies in C_2. Suppose, on the other hand, that there exists a rank 3 form $\beta \in \mathcal{A}$ whose radical does not lie on L. Then its quadric Q_β is tangent to each Q_t with $t \in T$ along the line connecting $P := \mathbb{P}\,\mathrm{rad}(\beta)$ and $\mathbb{P}\,\mathrm{rad}(\alpha_t)$; in particular, Q_β contains all lines connecting P with a dense subset of L (see Figure 3, right). The closure of the union of these lines—the projective plane spanned by L and P—is therefore contained in Q_β. Hence, the pre-image in V of this plane is a 3-dimensional β-isotropic space—but this contradicts the assumption that $\mathrm{rk}(\beta) = 3$.

Finally, we need to show that if $n \geq 5$ and $p \geq 3$, then the null-cone is *not* defined by the polarisations of det. Consider, to this end, the triple $(\alpha_1, \alpha_2, \alpha_3)$ of bilinear forms on $V = K^n$ such that the linear map associated to $s\alpha + t\beta + u\gamma$ relative to the orthogonal sum of the skew diagonal form $(.,.)$ on K^5 and the skew diagonal form on K^{n-5} equals

$$sA_1 + tA_2 + uA_3 = \left[\begin{array}{ccc|cc|c} s & t & 0 & 0 & 0 & \\ 0 & s & t & 0 & 0 & \\ \hline -u & 0 & 0 & t & 0 & \\ 0 & 2u & 0 & s & t & \\ 0 & 0 & -u & 0 & s & \\ \hline & & & & & sI_{n-5} \end{array}\right].$$

A direct computation shows that $\det(sA_1 + tA_2 + uA_3) = 0$. On the other hand, by Lemma 3.4 there does not exist a subspace U of K^n with $\dim(\sum_i A_i U) < \dim U$. We conclude that $(\alpha_1, \alpha_2, \alpha_3)$ is not nilpotent, and this concludes the proof of Theorem 3.2. □

4. SL(V) on skew-symmetric forms

Our results for skew-symmetric forms are similar to those for symmetric forms, except that the irreducible components of the null-cone become visible only on 3 or 4 copies. Recall that if $n := \dim(V)$ is odd, then all skew bilinear forms are singular and there are no invariants on one copy of $\bigwedge^2(V^*)$, so that the null-cone is the whole space. If n is even, then the invariant ring is generated by the Pfaffian (see, e.g. []), and the null-cone is irreducible.

Theorem 4.1. *The null-cone* $\mathrm{SL}(V)$ *on* $\bigwedge^2(V^*)^p$ *is equal to*

$$\{(\alpha_1, \ldots, \alpha_p) \mid \exists U \subseteq W \subseteq V \text{ with } \dim U + \dim W = n + 1 \text{ and } \alpha_i(U, W) = 0 \text{ for all } i = 1, \ldots, p\}.$$

Let $C_k^{(p)}$ *denote the subset of the null-cone where* U *can be chosen of dimension* $k(= 1, \ldots, \lceil \frac{n}{2} \rceil =: q)$. *Then the irreducible components of the null-cone are as follows.*

(1) *If* $n = 2q \geq 2$ *is even, then the null-cone on* $p = 2$ *copies is* $C_q^{(2)}$ *(hence irreducible), while the null-cone on* $p \geq 3$ *copies has precisely* q *components, namely* $C_k^{(p)}$ *for* $k = 1, \ldots, q$.
(2) *If* $n = 2q-1 \geq 3$ *is odd, then the null-cone on* $p = 2$ *copies is all of* $\bigwedge^2(V^*)^p$*; on* $p = 3$ *copies there are non-trivial invariants, and the components of the null-cone are precisely the* $C_k^{(3)}$ *with* $k \in \{1, 2, \ldots, q-4, q\}$ *(in particular, for* $n \leq 7$ *the null-cone is irreducible); on* $p = 4$ *copies the components of the null-cone are precisely the* $C_k^{(4)}$ *with* $k \in \{1, 2, \ldots, q-3, q\}$ *(in particular, for* $n \leq 5$ *the null-cone is irreducible); and on* $p \geq 5$ *copies the components of the null-cone are precisely the* $C_k^{(p)}$ *with* $k \in \{1, 2, \ldots, q-2, q\}$ *(in particular, for* $n \leq 3$ *the null-cone is irreducible).*

For the proof of this theorem we need a result from [8], which uses the following notation: $d(n, p)$ is the minimum, taken over all p-tuples $\alpha_1, \ldots, \alpha_p$ of skew bilinear forms on K^n, of the maximal dimension of a subspace that is isotropic with respect to all α_i. In other words, $d(n, p)$ is the maximal dimension of a common isotropic subspace of a generic p-tuple of skew bilinear forms on K^n.

Theorem 4.2 ([8, Main Theorem])**.** $d(n, p) = \lfloor \frac{2n+p}{p+2} \rfloor$.

Corollary 4.3. *For $n = 0, 2, 4, 6$ any triple of skew bilinear forms on K^n has a common isotropic subspace of dimension $n/2$. On the other hand, for all odd $n \geq 3$ and for all even $n \geq 8$ there exist triples $(\alpha_1, \alpha_2, \alpha_3)$ of skew bilinear forms on K^n for which there are* no *subspaces $0 \subsetneq U \subseteq W$ of K^n with $\dim U + \dim W = n$ and $\alpha_i(U, W) = 0$ for all i.*

Proof. The first statement is immediate from Theorem 4.2. Now let $n = 2q \geq 8$ be even, fix $k \in \{1, \ldots, q\}$, and suppose that for any triple $\alpha_1, \alpha_2, \alpha_3$ of skew bilinear forms on K^n there exist subspaces $0 \neq U \subseteq W$ of K^n with $\dim U = k = n - \dim W$ and $\alpha_i(U, W) = 0$ for all $i = 1, 2, 3$. The induced forms $\bar{\alpha}_i$, $i = 1, 2, 3$, on the space W/U of dimension $2(q - k)$ have a common isotropic subspace $U' \subseteq W/U$ of dimension $d(2(q - k), 3)$, by definition of the latter quantity. The pre-image of U' in W is then isotropic relative to all α_i and has dimension $d(2(q - k), 3) + k$. We thus find the inequality $d(2q, 3) \geq d(2(q - k), 3) + k$, which by Theorem 4.2 reads

$$\lfloor \frac{4q + 3}{5} \rfloor \geq \lfloor \frac{4(q - k) + 3}{5} \rfloor + k. \tag{6}$$

For $n = 2q = 8$, however, this inequality does not hold for any $k \in \{1, 2, 3, 4\}$. For $n = 2q = 10$ the only $k \in \{1, \ldots, 5\}$ for which it holds is $k = 1$, but it is easy to construct a triple of bilinear forms on K^{10} for which there are no U, W as above of dimensions $1, 9$—indeed, one can use for this the construction that follows.

Suppose that $n = 2q \geq 12$, and note that inequality (6) can only hold for $k \leq 5$. On the other hand, let $\alpha_1, \alpha_2, \alpha_3$ be the skew bilinear forms on K^n corresponding to the standard triple (A_1, A_2, A_3) of matrices satisfying

$$t_1 A_1 + t_2 A_2 + t_3 A_3 = \begin{bmatrix} t_2 & t_3 & & & & & & \\ t_1 & t_2 & \ddots & & & & & \\ & \ddots & \ddots & t_3 & & & & \\ & & t_1 & t_2 & & & & \\ & & & & -t_2 & -t_3 & & \\ & & & & -t_1 & \ddots & \ddots & \\ & & & & & \ddots & -t_2 & -t_3 \\ & & & & & & -t_1 & -t_2 \end{bmatrix}.$$

Using Lemma 3.4 one verifies that any subspace U of K^n satisfying $\dim(A_1 U + A_2 U + A_3 U) \leq \dim U$ has dimension $0, n/2$, or n. In particular, we should have $k \in \{0, q, n\}$—but we saw above that $1 \leq k \leq 5$, a contradiction.

We conclude that for $n = 2q \geq 8$ and fixed $k \in \{1, \ldots, q\}$ there exist triples $(\alpha_1, \alpha_2, \alpha_3)$ of skew bilinear forms on K^n for which there are no subspaces $U \subseteq W$ of K^n with $\dim U = k = n - \dim W$ and $\alpha_i(U, W) = 0$ for all i. As the non-existence of such a pair U, W with $\dim U = k$ is an open condition on the triple $(\alpha_1, \alpha_2, \alpha_3)$, there also exist triples for which there is no pair (U, W) with U of *any* dimension. This proves the corollary for even n.

For $n = 2q - 1 \geq 3$ odd we can construct $\alpha_1, \alpha_2, \alpha_3$ explicitly by a construction similar to that above: choose them corresponding to a standard triple (A_1, A_2, A_3)

of matrices satisfying

$$t_1A_1 + t_2A_2 + t_3A_3 = \begin{bmatrix} t_2 & t_3 & & & & & \\ t_1 & \ddots & \ddots & & & & \\ & \ddots & t_2 & t_3 & & & \\ & & t_1 & 0 & -t_3 & & \\ & & & -t_1 & -t_2 & \ddots & \\ & & & & \ddots & \ddots & -t_3 \\ & & & & & -t_1 & -t_2 \end{bmatrix}.$$

Using Lemma 3.4 one verifies that there are no subspaces $U \neq 0, K^n$ of K^n with $\dim(\sum_i A_iU) \leq \dim U$. □

Proof of Theorem 4.1. The description of the null-cone is proved in exactly the same way as for symmetric bilinear forms. We first prove that there are inclusions $C_k^{(p)} \subseteq C_q^{(p)}$ for the following values of the parameters:

1. (1) n arbitrary, k arbitrary, and $p = 2$;
2. (2) $n = 2q - 1 \geq 3$, $k = q - 1$, and p arbitrary;
3. (3) $n = 2q - 1 \geq 5$, $k = q - 2$, and $p \in \{3, 4\}$; or
4. (4) $n = 2q - 1 \geq 7$, $k = q - 3$, and $p = 3$.

These statements are proved as follows: let $(\alpha_1, \ldots, \alpha_p) \in C_k^{(p)}$ and let $U \subseteq W$ be a pair with $\dim U = k, \dim W = n - k + 1$, and $\alpha_i(U, W) = 0$ for all i. Then the α_i induce bilinear forms $\bar{\alpha}_i$ on the space W/U of dimension $n - 2k + 1$, and we find a subspace U' of W/U of dimension $d(n - 2k + 1, p)$ that is isotropic relative to all $\bar{\alpha}_i$. The pre-image of U' in W is then a space of dimension $d(n - 2k + 1, p) + k$ and isotropic relative to all α_i. Using Theorem 4.2 one finds that for the above values of the parameters this value $d(n - 2k + 1, p) + k$ is at least $\lfloor \frac{n}{2} \rfloor + 1$, which shows that $(\alpha_1, \ldots, \alpha_p) \in C_q^{(p)}$. This proves all inclusions above.

Now we prove that there are no other inclusions among the $C_k^{(p)}$ for other values of n, k, and p. Suppose first that $n = 2q$ is even, $p \geq 3$ and $k \in \{1, \ldots, q\}$. Then we find a p-tuple in $C_k^{(p)}$ not lying in any other $C_{k'}^{(p)}$ by a construction similar to the constructions in the proof of Corollary 4.3: Let $\alpha_1, \alpha_2, \alpha_3$ be forms with matrices A_1, A_2, A_3 for which $t_1A_1 + t_2A_2 + t_3A_3$ equals

$$\begin{bmatrix} t_2 & t_3 & & & & & & & & \\ t_1 & t_2 & t_3 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & t_1 & t_2 & t_3 & & & & & \\ & & & & & t_1A_1' + t_2A_2' + t_3A_3' & & & & \\ & & & & & & -t_3 & & & \\ & & & & & & -t_2 & \ddots & & \\ & & & & & & -t_1 & \ddots & -t_3 & \\ & & & & & & & \ddots & -t_2 & -t_3 \\ & & & & & & & & -t_1 & -t_2 \end{bmatrix}, \tag{7}$$

where the diagonal blocks have sizes $(k-1) \times k$, $(n-2k+1) \times (n-2k+1)$, and $k \times (k-1)$ from top left to bottom right, and where the A_i' are chosen such (skew relative to the skew diagonal) that they map no subspace $U \neq 0, K^{n-2k+1}$ of K^{n-2k+1} into a strictly smaller subspace; such A'_i exist by Corollary 4.3. Write $V_1 := \langle e_1, \ldots, e_k \rangle_K$, $V_2 := \langle e_{k+1}, \ldots, e_{n-k} \rangle_K$, and $V_3 := \langle e_{n-k+1}, \ldots, e_n \rangle_K$. Now suppose that U is a subspace of K^n for which $\dim \sum A_i U < \dim U$. Let $U_1 := U \cap V_1$, let U_2 be the projection of $U \cap (V_1 \oplus V_2)$ to V_2 along V_1, and let U_3 be the projection of U to V_3 along $V_1 \oplus V_2$. Then $\dim \sum_i A_i U_1 \geq \dim U_1$ unless $U_1 = V_1$, $\dim \sum_i A_i U_2 > \dim U_2$ unless $U_2 = 0$ or V_2, and $\dim \sum_i A_i U_3 > \dim U_3$ unless $U_3 = 0$. Summing up these dimensions, we find $\dim \sum_i A_i U < \dim U$ implies $U_1 = V_1$, $U_2 = 0$ or $U_2 = V_2$, and $U_3 = 0$. We conclude that $(V_1, V_1 \oplus V_2)$ is the only pair of subspaces $U \subseteq W$ with $\alpha_i(U, W) = 0$ and $\dim U + \dim W > n$. Hence $(\alpha_1, \alpha_2, \alpha_3)$ lie in $C_k^{(3)}$ but not in any other $C_{k'}^{(3)}$.

Next suppose that $n = 2q - 1 \geq 9$ is odd. Then we have to show that that $C_k^{(3)}$ for $k \notin \{q-1, q-2, q-3\}$ is *not* contained in any other $C_k^{(3)}$. This goes using a construction similar to that above for even n, choosing the A'_i—now square skew matrices of size $n - 2k + 1 = 2(q-k) \geq 8$—such that for all spaces U with $0 \subsetneq U \subsetneq K^{2(q-k)}$ we have $\dim A'_1 U + A'_2 U + A'_3 U > \dim U$; such matrices exist by Corollary 4.3.

Next, assuming $n = 2q - 1 \geq 7$, suppose that $p \geq 4$ and $k \in \{1, \ldots, q-3, q\}$. By writing down an appropriate standard quadruple of skew matrices $(A_1, \ldots, A_4)$ we show that $C_k^{(p)}$ is not contained in any other $C_{k'}^{(p)}$: take A_1, A_2, A_3, A_4 such that $\sum_i t_i A_i$ has the block shape of (7), where the outer two blocks are unchanged (i.e., A_4 has no non-zero entries there), but the inner block of size $2(q-k) \geq 6$ is as follows:

$$\begin{bmatrix}
t_2 & t_3 & t_4 & & & & & \\
0 & t_2 & \ddots & \ddots & & & & \\
t_1 & 0 & \ddots & t_3 & t_4 & & & \\
& \ddots & \ddots & t_2 & 0 & -t_4 & & \\
& & t_1 & 0 & -t_2 & -t_3 & \ddots & \\
& & & -t_1 & 0 & \ddots & \ddots & -t_4 \\
& & & & \ddots & \ddots & -t_2 & -t_3 \\
& & & & & -t_1 & 0 & -t_2
\end{bmatrix}$$

Again, applying Lemma 3.4, one readily verifies that this quadruple of skew matrices does not map any space U into a space of dimension $\leq \dim U$.

A similar construction for $n = 2q - 1 \geq 5$ with the following 4×4-block in the middle:

$$\begin{bmatrix}
t_3 & t_4 & t_5 & 0 \\
t_2 & t_3 & 0 & -t_5 \\
t_1 & 0 & -t_3 & -t_4 \\
0 & -t_1 & -t_2 & -t_3
\end{bmatrix}$$

shows that on $p \geq 5$ copies the set $C_{q-2}^{(p)}$ is not contained in any other $C_k^{(p)}$, either. □

Finally, we settle the question, for n even, of when the null-cone on p copies of $\bigwedge^2(V^*)$ is defined by the polarisations of the Pfaffian.

Theorem 4.4. *The null-cone* $\mathcal{N}(\bigwedge^2(V^*)^p)$ *with* $\dim V =: n$ *even is defined by the polarisations of the Pfaffian if and only if either* $p = 2$ *or* $n \in \{2, 4\}$.

Proof. The proof for $p = 2$ goes exactly as for symmetric bilinear forms, and for $n = 2$ the statement is trivial. Suppose therefore that $n = 4$, and let $\mathcal{A}$ be a vector space consisting of singular skew forms on K^4. We have to show that either the radicals of all forms in $\mathcal{A}$ intersect in a projective point, or there exist a line U and a plane $W \supseteq U$ in $\mathbb{P}^3$ with $\alpha(U, W) = 0$ for all $\alpha \in A$. By the statement for $p = 2$ we know that any *pair* of elements in $\mathcal{A}$ is of one of these two types.

We prove that in fact every pair $\alpha, \beta \in \mathcal{A}$ is of the first type. Indeed, take $\alpha, \beta \in \mathcal{A}$ non-zero (and hence of rank 2), suppose that $\operatorname{rad}\alpha$ and $\operatorname{rad}\beta$ are disjoint lines in $\mathbb{P}^3$, and let $U \subseteq W$ be a line and a plane in $\mathbb{P}^3$ such that $\alpha(U, W) = \beta(U, W) = 0$. For dimension reasons, U must intersect both $\operatorname{rad}\alpha$ and $\operatorname{rad}\beta$, and hence U is distinct from both of these lines. But then the α-orthoplement of U and the β-orthoplement of U are both planes containing W, and hence equal to W. On the other hand, the radicals of α and β are contained in the α-orthoplement and the β-orthoplement of U, respectively, hence in W. But this contradicts the assumption that the projective lines $\mathbb{P}\operatorname{rad}\alpha$ and $\mathbb{P}\operatorname{rad}\beta$ do not intersect.

We conclude that all radicals of elements in $\mathcal{A}$ intersect. But then they all lie in some plane W. Now if U is any line in W, then $\alpha(U, W) = 0$ for all $\alpha \in \mathcal{A}$, so that $\mathcal{A}$ 'lies in' C_2. This proves the theorem for $n = 4$.

Finally, for $n \geq 6$, we have to exhibit a triple of skew bilinear forms that is not nilpotent but whose span lies in the null-cone on $\bigwedge^2 V^*$. Choose for instance $\alpha_1, \alpha_2, \alpha_3$ with matrices A_1, A_2, A_3 such that

$$t_1 A_1 + t_2 A_2 + t_3 A_3 = \begin{bmatrix} t_2 & & & & & & & & \\ & \ddots & & & & & & & \\ & & t_2 & & & & & & \\ & & & t_2 & t_3 & 0 & & & \\ & & & t_1 & 0 & -t_3 & & & \\ & & & 0 & -t_1 & -t_2 & & & \\ & & & & & & -t_2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -t_2 \end{bmatrix}.$$

Using Lemma 3.4 one verifies that no subspace of K^n is mapped by all A_i into a strictly smaller subspace. This concludes the proof of the theorem. □

5. SL(V) on arbitrary bilinear forms

The invariants of $\mathrm{SL}(V)$ on $(V^* \otimes V^*)$ are known [1], but in contrast to the situation for linear maps and symmetric bilinear forms, it is not clear from them that the null-cone on one copy of $V^* \otimes V^*$ is irreducible. The following theorem states that it is, and also describes the components in several copies.

Theorem 5.1. *For* $p \geq 2$, *the null-cone of* $\mathrm{SL}(V)$ *on* $(V^* \otimes V^*)^p$ *has* $q := \lfloor \frac{n+1}{2} \rfloor$ *irreducible components given by*

$$C_k^{(p)} := \{(\alpha_1, \ldots, \alpha_p) \mid \exists U \subseteq W \subseteq V : \dim U = k, \dim W = n - k + 1, \text{ and } \alpha_i(U, W) = \alpha_i(W, U) = 0 \text{ for all } i = 1, \ldots, p\},\ k = 1, \ldots, q.$$

On $p = 1$ copy, the sets $C_k^{(1)}$ form a chain $C_1^{(1)} \subseteq C_2^{(1)} \subseteq \ldots \subseteq C_q^{(1)}$, and hence the null-cone equals the irreducible set $C_q^{(1)}$.

In the proof of this theorem we use the following lemma.

Lemma 5.2. *Let β be a symmetric form and γ a skew form on the vector space V of dimension ≥ 2. Then there exists a β-isotropic $v_0 \in V$ for which*

$$\dim\{v \in V \mid \beta(v_0, v) = \gamma(v_0, v) = 0\} \geq \dim V - 1$$

Proof. If the radical of γ has dimension ≥ 2, we may take for v_0 any β-isotropic vector in $\operatorname{rad}\gamma$. If $\operatorname{rad}\gamma$ has dimension 1 and is spanned by v_1, say, then there are two cases: either v_1 is β-isotropic and we may set $v_0 := v_1$, or $V = Kv_1 \oplus V'$, where $V' := v_1^{\perp_\beta}$. Then γ is non-degenerate on V' and if we find a v_0 in V' satisfying the conclusion of the lemma for V' instead of V, it also does the trick for V, as $\beta(v_1, v_0) = \gamma(v_1, v_0) = 0$.

Hence the case remains where γ is non-degenerate. Let B, C be the linear maps corresponding to β, γ relative to $(.,.)$ and choose any eigenvector v_0 of $C^{-1}B$. Then we have $Bv_0 \in KCv_0$ so that $\gamma(v, v_0)(= (v, Cv_0)) = 0$ implies $\beta(v, v_0)(= (v, Bv_0)) = 0$. In particular, v_0 is β-isotropic, and the vector space on the left-hand side in the lemma is the γ-orthoplement of v_0. □

Proof of Theorem 5.1. For the first statement, let $(\alpha_1, \ldots, \alpha_p)$ be a nilpotent p-tuple of bilinear forms and write $\alpha_i = \beta_i + \gamma_i$ for all i, with β_i symmetric and γ_i skew. Let B_i, C_i be the linear maps associated β_i, γ_i, respectively. By assumption there exists a one-parameter subgroup $\lambda : K^* \to \mathrm{SL}(V)$ with $\lim_{t\to 0} \lambda(t)\alpha_i = 0$ for all i. But this implies that also $\lambda(t)\beta_i, \lambda(t)\gamma_i \to 0$ for $t \to 0$. *A fortiori*, the $2p$-tuple $(B_1, \ldots, B_p, C_1, \ldots, C_p)$ is nilpotent under the larger group $\mathrm{SL}(V) \times \mathrm{SL}(V)$, and by Theorem 2.1 there exist subspaces $U', U'' \subseteq V$ of dimensions k and $k-1$ such that $B_iU', C_iU' \subseteq U''$ for all i. Let W' be the orthoplement of U' relative to our fixed form $(.,.)$, set $U := U' \cap W'$ and $W := W' + U'$. Then $U \subseteq W$, $\dim U + \dim W = n + 1$, and $\beta_i(U, W) = \gamma_i(U, W) = 0$. But then also $\alpha_i(U, W) = \alpha_i(W, U) = 0$, as claimed.

Now we prove $C_k^{(1)} \subseteq C_{k+1}^{(1)}$ for $k < q$. To this end, let $U \subseteq W$ be subspaces of V with $\dim U + \dim W = n + 1$. We want to prove that a form $\alpha \in V^* \otimes V^*$ lying in $C_k^{(1)}$ by virtue of $\alpha(U, W) = \alpha(W, U) = 0$ also lies in $C_{k+1}^{(1)}$. Indeed, write $\alpha = \beta + \gamma$, where β is symmetric and γ is skew. The forms β, γ induce forms $\bar{\beta}, \bar{\gamma}$ of the same signature on W/U, and by the preceding lemma there exists a $\bar{w}_0 \in W/U$ for which

$$\dim\{\bar{w} \in W/U \mid \bar{\beta}(\bar{w}, \bar{w}_0) = \bar{\gamma}(\bar{w}, \bar{w}_0) = 0\} \geq \dim W/U - 1.$$

Let w_0 be a pre-image of $\bar{w}$ in W, set $U' := U \oplus Kw_0$, and let $W' \subseteq W$ be a subspace of codimension 1 that contains w_0 and whose image in W/U is contained in the space above. Then we still have $\alpha(U', W') = 0$ and $\dim U' + \dim W' = n + 1$, but now $\dim U' = k + 1$, as claimed.

Finally, we have two show that on $p \geq 2$ copies there are no inclusions among the sets $C^{(k)}$ with $k = 1, \ldots, q$ are distinct. But their intersections with the set of p-tuples of *symmetric* bilinear forms are already distinct, see Theorem 3.1. □

The last question to be answered here is whether the polarisations of the invariants on one copy of $V^* \otimes V^*$ define the null-cone on more copies. The answer can be deduced from the answers for symmetric forms and for skew forms.

Theorem 5.3. *The null-cone of* $\mathrm{SL}(V)$ *on* $(V^* \otimes V^*)^p$ *is defined by the polarisations to* $p \geq 2$ *copies of the invariants on* $V^* \otimes V^*$ *if and only if* $\dim V \leq 2$.

Proof. For $\dim V = 1$ the statement is trivial. Suppose that $\dim V = 2$ and let $\mathcal{A}$ be a space of nilpotent bilinear forms on V. If $\alpha \in \mathcal{A}$, then by theorem 5.1 both the symmetric and the skew component of α is singular. As the skew component has even rank, it is then zero. Hence $\mathcal{A}$ consists of symmetric forms only, and therefore the existence of a common radical for forms in $\mathcal{A}$ follows from Theorem 3.2.

Suppose now that $n \geq 3$. Let $\beta_1, \beta_2, \gamma_1$ be the bilinear forms on K^n whose matrices B_1, B_2, C_1 relative to the orthogonal sum $(.,.)$ of the skew diagonal forms on K^3 and K^{n-3} satisfy

$$s_1 B_1 + s_2 B_2 + t_1 C_1 = \left[\begin{array}{ccc|c} s_1 & s_2 & 0 & \\ t_1 & 0 & s_2 & \\ 0 & -t_1 & s_1 & \\ \hline & & & sI_{n-4} \end{array}\right].$$

A direct computation shows that $\det(s_1 B_1 + s_2 B_2 + t_1 C_1)$ is identically zero. We claim that actually $\mathcal{A} := \langle \beta_1, \beta_2, \gamma_1 \rangle_K$ consists entirely of nilpotent bilinear forms; as the determinant is not the only invariant, the preceding computation does not prove this yet. But let α be in $\mathcal{A}$ with matrix A. Then A^t—where transposition, as always, is relative to the form $(.,.)$—defines the form α^t, which by the definition of $\mathcal{A}$ also lies in $\mathcal{A}$ and the singular matrix pencil $\langle A, A^t \rangle_K$ has a subspace U of K^n for which $W' := A^t U + AU$ has dimension $< \dim U$. But then the orthoplement W of W' relative to $(.,.)$ is a subspace of K^n of dimension $> n - \dim U$ satisfying $\alpha(W, U) = \alpha^t(W, U) (= \alpha(U, W)) = 0$. Replacing (U, W) by the pair $(U \cap W, U + W)$ as usual, we find a witness for the nilpotency of α.

However, the pair $(\beta_1 + \gamma_1, \beta_2)$ of bilinear forms is not nilpotent. Indeed, if it were, then there would be $U \subseteq W$ with $\dim U + \dim W = n + 1$ and $\beta_1(U, W) = \beta_2(U, W) = \gamma_1(U, W) = 0$, i.e., with $\dim B_1 U + B_2 U + C_1 U < \dim U$. By Lemma 3.4 no U with this property exists. □

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