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Constructing Lie Algebras of First Order Differential Operators

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Abstract

We extend Guillemin and Sternberg's Realization Theorem for transitive Lie algebras of formal vector fields to certain Lie algebras of formal first order differential operators, and show that Blattner's proof of the Realization Theorem allows for a computer implementation that automatically reproduces many realizations derived in the existing literature, and that can also be used to compute new realizations. Applications include the explicit construction of quasi-exactly solvable Hamiltonians, and of finite-dimensional irreducible modules over semisimple Lie algebras.

1. Introduction

For fixed $\lambda \in \mathbb{R}$, the space

$$\mathfrak{g}_\lambda := \langle \partial_x,\ -2x\partial_x + \lambda, -x^2\partial_x + \lambda x\rangle_{\mathbb{R}}$$

of first order differential operators on the real line form a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. If λ is a non-negative integer, then the finite-dimensional vector space

$$V_\lambda := \langle 1, x, x^2, \ldots, x^\lambda\rangle_{\mathbb{R}}$$

is invariant under $\mathfrak{g}_\lambda$, whence under any element of the latter's enveloping associative algebra. Up to coordinate changes and gauge transforms, many physically meaningful Hamiltonians H on the real line are quadratic elements of this enveloping algebra for certain $\lambda \in \mathbb{N}$; we find that for such H a finite part of the spectrum of the associated Schrödinger equation $H\psi = E\psi$ can be determined, namely the part corresponding to V_λ. For this reason, such Hamiltonians are called *quasi-exactly solvable* (8).

One step towards finding quasi-exactly solvable Hamiltonians in n dimensions consists of computing Lie algebras of first order differential operators in n variables; this is the goal of the paper at hand. We will restrict our attention to a particular type of Lie algebras of differential operators, which can be formally described as follows. Let K be a field of characteristic 0, let $\mathfrak{g}$ be a Lie algebra over K, and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ of finite codimension n. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a list of variables, denote by $K[[\mathbf{x}]]$ the K-algebra of formal power series in the x_i with coefficients from K, and write $\hat{\mathfrak{D}}$ for the Lie algebra of K-derivations of $K[[\mathbf{x}]]$. Any element of $\hat{\mathfrak{D}}$ is of the form $\sum_i f_i \partial_i$, where ∂_i denotes differentiation with respect to x_i, and the f_i are elements of $K[[\mathbf{x}]]$, called the *coefficients* of the derivation. Let $\hat{\mathfrak{D}}_0$ denote the *isotropy subalgebra* at the origin, i.e., the algebra consisting of those derivations that leave the maximal ideal of $K[[\mathbf{x}]]$ invariant. Now a *realization of* $(\mathfrak{g}, \mathfrak{k})$ *in terms of derivations* is by definition a homomorphism $\phi : \mathfrak{g} \to \hat{\mathfrak{D}}$ with the property that $\phi^{-1}(\hat{\mathfrak{D}}_0) = \mathfrak{k}$. As $\operatorname{codim}_{\mathfrak{g}} \mathfrak{k} = \operatorname{codim}_{\hat{\mathfrak{D}}} \hat{\mathfrak{D}}_0 = n$, the image $\phi(\mathfrak{g})$ is a *transitive* Lie algebra, that is: $\phi(\mathfrak{g}) \cap (\partial_i + \hat{\mathfrak{D}}) \neq \emptyset$ for all $i = 1, \ldots, n$. The Realization Theorem of Guillemin and Sternberg states that a realization in terms of derivations always exists, and that it is unique up to formal coordinate changes (13).

The space $\hat{\mathfrak{D}} + K[[\mathbf{x}]]$ is a Lie algebra with respect to the Lie bracket defined by

$$[X + f, Y + g] := [X, Y] + X(g) - Y(f),\ X, Y \in \hat{\mathfrak{D}}, f, g \in K[[\mathbf{x}]];$$

its elements are called (formal) *first order differential operators* (in n variables). A *realization of* $(\mathfrak{g}, \mathfrak{k})$ *in terms of first order differential operators* is by definition a homomorphism $\psi : \mathfrak{g} \to \hat{\mathfrak{D}} + K[[\mathbf{x}]]$ satisfying

$$\psi(X) = \phi(X) + c(X),\ X \in \mathfrak{g}$$

for some realization ϕ of $(\mathfrak{g}, \mathfrak{k})$ in terms of derivations and some linear map $c : \mathfrak{g} \to K[[\mathbf{x}]]$. Given ϕ and c, the map ψ above is a homomorphism of Lie algebras if and only if

$$c([X, Y]) = \phi(X)(c(Y)) - \phi(Y)(c(X)),$$

i.e., if and only if c is a cocycle of $\mathfrak{g}$ with values in $K[[\mathbf{x}]]$, where the latter is viewed as a $\mathfrak{g}$-module through ϕ. The coboundaries are of the form $c : X \mapsto X(f)$ for a fixed $f \in K[[\mathbf{x}]]$; adding such a coboundary to a realization in terms of differential operators is called an *infinitesimal gauge transformation.*

We conclude that, in order to construct 'all' realizations of $(\mathfrak{g}, \mathfrak{k})$ in terms of first order differential operators, it suffices to construct a realization ϕ of $(\mathfrak{g}, \mathfrak{k})$ in terms of derivations (Section 2) and to describe the cohomology group $H^1(\mathfrak{g}, (K[[\mathbf{x}]], \phi))$ explicitly (Section 3). This approach is widely spread in the literature (7; 16; 18). Indeed, (18) describes realizations (in terms of first order differential operators) of pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ is a Borel subalgebra, and concludes with the question of whether the approach followed in that paper can be generalized to the situation where $\mathfrak{g}$ is not semisimple or $\mathfrak{k}$ is not a Borel subalgebra. The paper at hand shows that it can; in particular, Section 4 contains a variety of realizations computed in GAP (6). In fact, the images of all realizations computed there consist of *polynomial* first order differential operators; such a realization is a called a *polynomial realization.*

2. Realization in terms of Vector Fields

To formulate an explicit realization, let $\mathfrak{g}$ be a Lie algebra over a field K of characteristic zero, and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ of finite codimension n. Choose a well-ordered basis $\mathcal{C}$ of $\mathfrak{k}$, and let $\mathbf{Y} := (Y_1, \ldots, Y_n)$ be a list of elements of $\mathfrak{g}$ projecting onto a basis of $\mathfrak{g}/\mathfrak{k}$. Order $\mathcal{B} := \mathcal{C} \cup \{Y_1, \ldots, Y_n\}$ as follows: $X \leq Y_i$ for all $X \in \mathcal{C}$ and $i = 1, \ldots, n$, and $Y_i \leq Y_j :\Leftrightarrow i \leq j$. Consider the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ with PBW-basis corresponding to the ordered basis $\mathcal{B}$, and let $\chi_i : U(\mathfrak{g}) \to K$ $(i = 1, \ldots, n)$ map u to the coefficient of the PBW-monomial Y_i in u.

Theorem 2.1 (Realization Formula): *The map $\phi_{\mathbf{Y}} : \mathfrak{g} \to \hat{\mathfrak{D}}$ defined by*

$$\phi_{\mathbf{Y}}(X) := \sum_{i=1}^{n} \left(\sum_{\mathbf{m} \in \mathbb{N}^n} \chi_i(\mathbf{Y}^{\mathbf{m}} X) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} \right) \partial_i,$$

is a transitive realization of the pair $(\mathfrak{g}, \mathfrak{k})$; here $\mathbf{Y}^{\mathbf{m}} := Y_1^{m_1} \cdots Y_n^{m_n}$, $\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$, and $\mathbf{m}! := m_1! \cdots m_n!$

Kantor presents an equally general realization formula in (14), and it would be interesting to investigate the relation between the two. Here, we choose to work with the formula above because of its computational transparency and the fact that it can easily be modified to a realization formula in terms of first order differential operators. We briefly sketch a proof of our Realization Formula, as we will need some of its ingredients in Section 3. For details see (1) and (5).

Proof: Following (1), we consider the $\mathfrak{g}$-module

$$A := \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), K),$$

which is defined as follows: view $U(\mathfrak{g})$ as a left $U(\mathfrak{k})$-module, and endow K with the trivial left $U(\mathfrak{k})$-module structure. Then A is the space of all homomorphisms $U(\mathfrak{g}) \to K$ of $U(\mathfrak{k})$-modules. The action of $\mathfrak{g}$ on A is defined by

$$(Xa)u := a(uX), \text{ for } X \in \mathfrak{g}, a \in A, \text{ and } u \in U(\mathfrak{g}).$$

Blattner endows A with the structure of a commutative algebra, with respect to which $\mathfrak{g}$ turns out to act by derivations.

By the PBW-theorem $U(\mathfrak{g})$ is a free $U(\mathfrak{k})$-module with basis $\{\mathbf{Y}^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^n\}$.

An element of A is therefore determined by its values on these monomials, and these values can be prescribed arbitrarily. From this it follows that the pull-back α^* of the K-linear map $\alpha : K[\mathbf{x}] \to U(\mathfrak{g})$ determined by $\alpha(\mathbf{x}^\mathbf{m}) = \mathbf{Y}^\mathbf{m}$ is a linear isomorphism from A onto $K[\mathbf{x}]^*$. For $\mathbf{m} \in \mathbb{N}^n$ let $f_\mathbf{m} \in K[\mathbf{x}]^*$ be the element determined by $\langle x^\mathbf{r}, f_\mathbf{m}\rangle = \delta_{\mathbf{r},\mathbf{m}}$, $\mathbf{r} \in \mathbb{N}^n$. Then the map $\beta : K[\mathbf{x}]^* \to K[[\mathbf{x}]]$ defined by

$$\beta\left(\sum_{\mathbf{m}\in\mathbb{N}^n} \gamma_\mathbf{m} f_\mathbf{m}\right) = \sum_{\mathbf{m}\in\mathbb{N}^n} \gamma_\mathbf{m} \frac{\mathbf{x}^\mathbf{m}}{\mathbf{m}!}$$

is a linear isomorphism, and it turns out that $\beta \circ \alpha^*$ is an algebra isomorphism $A \to K[[\mathbf{x}]]$. Carrying over the action of $\mathfrak{g}$ on A to an action of $\mathfrak{g}$ on $K[[\mathbf{x}]]$ by means of this isomorphism, we find the Realization Formula. □

One readily verifies that the Realization Formula only depends on $\mathbf{Y}$, not on $\mathcal{C}$; this justifies the notation $\phi_\mathbf{Y}$. The computational interpretation of $\phi_\mathbf{Y}$ is as follows: to compute the coefficient of $\mathbf{x}^\mathbf{m}$ in the coefficient of ∂_i in $\phi_\mathbf{Y}(X)$, simply multiply $\mathbf{Y}^\mathbf{m}$ from the right by X, reduce this element of $U(\mathfrak{g})$ to PBW-normal form, extract the coefficient of Y_i, and divide by $\mathbf{m}$! This algorithm was implemented in `GAP` (6), and has proved an effective tool in computing explicit realizations of pairs $(\mathfrak{g}, \mathfrak{k})$. The following proposition is of particular interest.

Proposition 2.1: *Let $\mathfrak{m}$ be a vector space complement of $\mathfrak{k}$ in $\mathfrak{g}$, and assume that $\mathfrak{m}$ is in fact a subalgebra of $\mathfrak{g}$ acting locally nilpotently on the latter, that is: we have*

$$\forall X \in \mathfrak{g}\ \exists d \in \mathbb{N} : (\operatorname{ad}_\mathfrak{g} \mathfrak{m})^d X = 0.$$

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a basis of $\mathfrak{m}$ satisfying $[\mathfrak{m}, Y_i] \subseteq \langle Y_{i+1}, \ldots, Y_n\rangle_K$ for all i. Let σ be any permutation of $\{1, \ldots, n\}$, and write $\mathbf{Y}_\sigma := (Y_{\sigma(1)}, \ldots, Y_{\sigma(n)})$. Then $\phi_{\mathbf{Y}_\sigma}$ is a polynomial realization. Indeed, if $\operatorname{ad}_\mathfrak{g}(\mathfrak{m})^d X = 0$, then the coefficients of $\phi_{\mathbf{Y}_\sigma}(X)$ contain only monomials of total degree $< d$.

The proof of (5, Theorem 1.7) is easily modified to a proof of this proposition, and as that proof requires rather technical notation not to be used elsewhere in the present paper, we omit it here. We do, however, want to stress the following subtlety: it is tempting to think, in the setting of Proposition 2.1, that $\phi_\mathbf{Y}$ is polynomial for *any* basis $\mathbf{Y}$ of $\mathfrak{m}$. The following example shows that this is not true.

Example: Let $\mathfrak{g} = \langle Y_1, Y_2, Y_3\rangle_K$ be a Lie algebra with Lie bracket determined by

$$[Y_1, Y_2] = -Y_1 + Y_3,\ [Y_1, Y_3] = 0,\ \text{and } [Y_2, Y_3] = Y_1 - Y_3.$$

Then $\mathfrak{g}$ is nilpotent, but the realization $\phi_\mathbf{Y}$ of $(\mathfrak{g}, 0)$ is not nilpotent. Indeed, by induction on k one can simultaneously show that

$$Y_2^k Y_1 = Y_1 Y_2^k + \sum_{i=0}^{k-1} k(k-1)\cdots(i+1)(Y_1 Y_2^i - Y_2^i Y_3) \text{ and}$$
$$Y_3 Y_2^k = Y_2^k Y_3 + \sum_{i=0}^{k-1} k(k-1)\cdots(i+1)(-Y_1 Y_2^i + Y_2^i Y_3).$$

The first of these equalities shows that the coefficient of Y_1 in $Y_2^k Y_1$ is equal to $k!$ for all k, so that the term $x_2^k \partial_1$ occurs with coefficient 1 in $\phi_{\mathbf{Y}}(Y_1)$.

We will apply Proposition 2.1 in the following setting: let $\mathfrak{g}$ be a finite-dimensional split semisimple Lie algebra, and let $\mathfrak{h}$ be a split Cartan subalgebra of $\mathfrak{g}$. Choose a fundamental subset Π in the root system $\Delta \subseteq \mathfrak{h}^*$, and denote by $\Delta_\pm$ the corresponding sets of positive and negative roots, respectively. Let Π_0 be a subset of Π, and denote by Δ_0 the intersection of the $\mathbb{Z}$-span of Π_0 with Δ. Set

$$\mathfrak{g}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_0} \mathfrak{g}_\alpha,\ \mathfrak{u}_\pm := \bigoplus_{\alpha\in\Delta_\pm\setminus\Delta_0} \mathfrak{g}_\alpha,\ \text{and } \mathfrak{p}_\pm := \mathfrak{g}_0 \oplus \mathfrak{u}_\pm.$$

Then $\mathfrak{u}_+$ is a nilpotently acting subalgebra complementary to the parabolic subalgebra $\mathfrak{p}_-$ of $\mathfrak{g}$, and if $\mathbf{Y}$ is a basis of $\mathfrak{u}_+$ consisting of $\mathfrak{h}$-root vectors, then $\phi_{\mathbf{Y}}$ is a polynomial realization of $(\mathfrak{g}, \mathfrak{p}_-)$. Polynomial realizations of such semisimple-parabolic pairs are also computed in (11), (12), (17), and (19). Section 4 contains some examples handled by our `GAP`-program based on the Realization Formula.

3. Realization in terms of First Order Differential Operators

The main ingredient in Blattner's proof of the Realization Theorem is the $\mathfrak{g}$-module $\mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), K) =: A$, which serves as a model for the algebra of formal power series in $\mathrm{codim}_{\mathfrak{g}}(\mathfrak{k})$ variables if this codimension is finite. To extend Blattner's construction to realizations in terms of first order differential operators, we must compute $H^1(\mathfrak{g}, A)$, as argued in Section 1. In this section we prove the following theorem, which is the formal analogue of (16, Theorem 8.3).

THEOREM 3.1: *The cohomology space $H^1(\mathfrak{g}, A)$ is canonically isomorphic to $(\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}])^*$.*

A consequence of this theorem and its proof will be an explicit realization in terms of first order differential operators. To formulate this realization, let $\mathfrak{g}, \mathfrak{k}, \mathcal{C}, \mathbf{Y}, \mathcal{B}$, and χ_i be as in Section 2, and let $\pi_{\mathfrak{k}} : U(\mathfrak{g}) \to \mathfrak{k}$ be the projection onto $\mathfrak{k}$ along the space spanned by $\mathbf{Y}$ and the PBW-monomials of degree ≥ 2.

Theorem 3.2: *Let* $\eta \in \mathfrak{k}^*$ *be such that* $\eta([\mathfrak{k}, \mathfrak{k}]) = 0$. *Then the map*

$$\phi_{\mathbf{Y},\eta}(X) = \phi_{\mathbf{Y}}(X) + \sum_{\mathbf{m} \in \mathbb{N}^n} \eta(\pi_{\mathfrak{k}}(\mathbf{Y}^{\mathbf{m}} X)) \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}$$

is a realization of $(\mathfrak{g}, \mathfrak{k})$ *in terms of first order differential operators. Moreover, if* ϕ *is any such realization, then there exists a unique* $\eta \in (\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}])^*$ *such that* $\phi_{\mathbf{Y},\eta}$ *is equivalent to* ϕ *under a coordinate transformation followed by an infinitesimal gauge transformation.*

In Theorem 3.1 we do not assume that $\mathfrak{k}$ has finite codimension in $\mathfrak{g}$. In fact, this condition is not essential in Theorems 2.1 and 3.2 either, but there we do assume it to keep notation ($\mathbf{Y}^{\mathbf{m}}$) and interpretation (formal power series in a finite rather than infinite number of variables) transparent. We proceed to compute the space of cocycles of $\mathfrak{g}$ with values in A.

Lemma 3.1: *Let* $\eta \in (U(\mathfrak{g})\mathfrak{g})^*$ *be such that* $\eta(\mathfrak{k}U(\mathfrak{g})\mathfrak{g}) = 0$. *Then the map* $c_\eta : \mathfrak{g} \to U(\mathfrak{g})^*$ *defined by*

$$c_\eta(X)u := \eta(uX) \text{ for } X \in \mathfrak{g}, u \in U(\mathfrak{g})$$

is an element of $Z^1(\mathfrak{g}, A)$.

Proof: The condition on η ensures that $c_\eta(X)(\mathfrak{k}U(\mathfrak{g})) = 0$, so that the image of c_η is contained in A. To check that c_η is a cocycle, let $X, Y \in \mathfrak{g}$ and $u \in U(\mathfrak{g})$, and compute

$$\begin{aligned} c_\eta([X,Y])u &= \eta(uXY) - \eta(uYX) \\ &= c_\eta(Y)(uX) - c_\eta(X)(uY) \\ &= (Xc_\eta(Y))(u) - (Yc_\eta(X))(u), \end{aligned}$$

where the last equality follows from the definition of the $\mathfrak{g}$-module structure on A. We conclude that $c_\eta([X,Y]) = Xc_\eta(Y) - Yc_\eta(X)$, whence the lemma. □

Lemma 3.2: *Every cocycle of* $\mathfrak{g}$ *with values in* A *is of the form* c_η *for some* η *as in Lemma 3.1.*

For the proof of this lemma and Lemma 3.5 we will use the following set-theoretic lemma, whose proof is standard and therefore omitted.

Lemma 3.3: *Let* $\mathcal{S}$ *be a well-ordered set. Then there exists a unique well-order on the collection* $\mathcal{C}$ *of finite subsets of* $\mathcal{S}$ *with the following properties:* $\emptyset \leq S$ *for all* $S \in \mathcal{C}$, *and if* $S, T \in \mathcal{C}$ *are both non-empty, then*

$$S \leq T \Leftrightarrow$$
$$\max(S) < \max(T) \text{ or } (\max(S) = \max(T) \text{ and } (S \setminus \max(S)) \leq (T \setminus \max(T))).$$

Proof (Proof of Lemma 3.2): Let $c \in Z^1(\mathfrak{g}, A)$, and consider the linear function $\tilde{\eta}$ on $U(\mathfrak{g}) \otimes \mathfrak{g}$ determined by

$$\tilde{\eta}(u \otimes X) = c(X)u \text{ for } u \in U(\mathfrak{g}), X \in \mathfrak{g}.$$

We want to show that $\tilde{\eta}$ factorises into $\eta \circ \pi$, where π is the multiplication map $U(\mathfrak{g}) \otimes \mathfrak{g} \to U(\mathfrak{g})\mathfrak{g}$ and η is an element of $(U(\mathfrak{g})\mathfrak{g})^*$. This is equivalent to showing that $\ker \pi$ is contained in $\ker \tilde{\eta}$.

To this end, let $\mathcal{B}$ be any well-ordered basis of $\mathfrak{g}$, and let

$$\mathcal{M} := \{X_1 X_2 \cdots X_d \mid X_1, \ldots, X_d \in \mathcal{B} \text{ and } X_1 \leq X_2 \leq \ldots \leq X_d\}$$

be the corresponding set of PBW-monomials. This set is also well-ordered, as follows: if $\mathbf{X} = (X_1, \ldots, X_d)$ and $\mathbf{Y} = (Y_1, \ldots, Y_e)$ are non-decreasing lists of elements of $\mathcal{B}$, then $X_1 \cdots X_d \leq Y_1 \cdots Y_e$ if and only if either $d < e$, or $d = e$ and $\mathbf{X}$ is lexicographically at most $\mathbf{Y}$, that is: either $\mathbf{X} = \mathbf{Y}$ or if i is the smallest index for which $X_i \neq Y_i$, then $X_i < Y_i$. Next, we endow the Cartesian product $\mathcal{M} \times \mathcal{B}$ with the lexicographic well-order

$$(u, X) \leq (v, Y) :\Leftrightarrow u < b \text{ or } (u = v \text{ and } X \leq Y),\ u, v \in \mathcal{M},\ X, Y \in \mathcal{B}.$$

The collection of finite subsets of $\mathcal{M} \times \mathcal{B}$, in turn, is well-ordered as in Lemma 3.3. To an element

$$r = \sum_{u \in \mathcal{M}, X \in \mathcal{B}} \gamma_{u,X} u \otimes X \in U(\mathfrak{g}) \otimes \mathfrak{g},$$

we associate the finite subset

$$\operatorname{supp}(r) := \{(u, X) \in \mathcal{M} \times \mathcal{B} \mid \gamma_{u,X} \neq 0\}$$

of $\mathcal{M} \times \mathcal{B}$, called the *support* of r. To show that $\pi(r) = 0$ implies $\tilde{\eta}(r) = 0$, we proceed by induction on $\operatorname{supp}(r)$. Let $r \neq 0$ as above be an element of the kernel of π, and assume that all elements $s \in \ker \pi$ with support smaller than $\operatorname{supp}(r)$ are in the kernel of $\tilde{\eta}$. By the PBW-theorem there exists a pair $(u, X) \in \operatorname{supp}(r)$ for which u ends on $Y > X$. Then, writing $u = u'Y$ for some $u' \in \mathcal{M}$ of degree $\deg(u) - 1$, we have

$$\begin{aligned} 0 &= \pi(r - \gamma_{u,X}(u'Y) \otimes X) + \gamma_{u,X} u'YX \\ &= \pi(r - \gamma_{u,X}(u'Y) \otimes X) + \gamma_{u,X}(u'XY + u'[Y, X]) \\ &= \pi(s), \end{aligned}$$

where

$$s := r - \gamma_{u,X} u'Y \otimes X + \gamma_{u,X} u'X \otimes Y + \gamma_{u,X} u' \otimes [Y, X].$$

Now u' is smaller than $u'Y$, and $u'X$, too, contains only PBW-monomials smaller than $u'Y$. This implies that all elements in the supports of $u' \otimes [Y, X]$ and $u'X \otimes Y$

are smaller than (u, X), so that $\mathrm{supp}(s) < \mathrm{supp}(r)$ and $\tilde{\eta}(s) = 0$ by the induction hypothesis. Using the $\mathfrak{g}$-module structure on A and the fact that c is a cocycle we find

$$\begin{aligned}\tilde{\eta}(u' \otimes [Y, X]) &= c([Y, X])u' \\ &= (Yc(X) - Xc(Y))u' \\ &= c(X)(u'Y) - c(Y)(u'X) \\ &= \tilde{\eta}(u'Y \otimes X - u'X \otimes Y).\end{aligned}$$

Combining this with $\tilde{\eta}(s) = 0$ we find $\tilde{\eta}(r) = 0$, as claimed. This concludes the proof that $\tilde{\eta}$ factorises into $\eta \circ \pi$. By construction, $\eta \in (U(\mathfrak{g})\mathfrak{g})^*$ satisfies $\eta(uX) = c(X)u$ for all $u \in U(\mathfrak{g})$ and $X \in \mathfrak{g}$, so that

$$\eta(\mathfrak{k}U(\mathfrak{g})\mathfrak{g}) = C(\mathfrak{g})(\mathfrak{k}U(\mathfrak{g})) = 0,$$

and $c = c_\eta$. □

We have now proved that the map

$$(U(\mathfrak{g})\mathfrak{g}/\mathfrak{k}U(\mathfrak{g})\mathfrak{g})^* \to Z^1(\mathfrak{g}, A),\ \eta \mapsto c_\eta$$

is surjective. As it is also injective, the space on the left-hand side parameterizes $Z^1(\mathfrak{g}, A)$. Let us characterise the coboundaries in a similar fashion.

Lemma 3.4: *We have*

$$B^1(\mathfrak{g}, A) = \{c_\eta \mid \eta \in (U(\mathfrak{g})\mathfrak{g})^*,\ \eta(\mathfrak{k}U(\mathfrak{g})) = 0\}.$$

Proof: Let $a \in A$, and consider the corresponding coboundary

$$\delta(a)X := Xa,\ \text{for } X \in \mathfrak{g}.$$

Compute

$$(\delta(a)X)u = (Xa)(u) = a(uX),$$

so if we set $\eta := a|_{U(\mathfrak{g})\mathfrak{g}}$, then $\delta(a) = c_\eta$. □

Combining Lemmas 3.2 and 3.4, we find that $H^1(\mathfrak{g}, A) = Z^1(\mathfrak{g}, A)/B^1(\mathfrak{g}, A)$ is parameterized by

$$(U(\mathfrak{g})\mathfrak{g}/\mathfrak{k}U(\mathfrak{g})\mathfrak{g})^*/(U(\mathfrak{g})\mathfrak{g}/\mathfrak{k}U(\mathfrak{g}))^*,$$

which is canonically isomorphic to

$$(\mathfrak{k}U(\mathfrak{g})/\mathfrak{k}U(\mathfrak{g})\mathfrak{g})^*.$$

The following lemma exposes this space as the space of 1-dimensional characters of $\mathfrak{k}$.

Lemma 3.5: *We have*

$$\mathfrak{k} + \mathfrak{k}U(\mathfrak{g})\mathfrak{g} = \mathfrak{k}U(\mathfrak{g}) \text{ and}$$
$$\mathfrak{k} \cap \mathfrak{k}U(\mathfrak{g})\mathfrak{g} = [\mathfrak{k}, \mathfrak{k}],$$

so that $\mathfrak{k}U(\mathfrak{g})/\mathfrak{k}U(\mathfrak{g})\mathfrak{g}$ *is canonically isomorphic to* $\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}]$.

Proof: The first statement is clear from

$$U(\mathfrak{g}) = K \cdot 1 + U(\mathfrak{g})\mathfrak{g}.$$

In the second statement, the inclusion $\supseteq$ is clear. For the converse, let $\mathcal{B}$ be a basis of $\mathfrak{g}$ such that $\mathcal{C} := \mathcal{B} \cap \mathfrak{k}$ is a basis of $\mathfrak{k}$, and choose a well-order on $\mathcal{B}$ such that all elements of $\mathcal{C}$ are smaller than all elements of $\mathcal{B} \setminus \mathcal{C}$. Let $\mathcal{M}'$ be the set of PBW-monomials with respect to $\mathcal{B}$, of degree > 0. We equip $\mathcal{C} \times \mathcal{M}'$ with the following well-order:

$$\begin{aligned}(X, u) \leq (Y, v) := &\deg(u) < \deg(v) \text{ or}\\ &(\deg(u) = \deg(v) \text{ and } X < Y) \text{ or}\\ &(\deg(u) = \deg(v) \text{ and } X = Y \text{ and } u \leq v),\end{aligned}$$

where the last alternative refers to the order on $\mathcal{M}$ introduced in the proof of Lemma 3.2. Again, we endow the collection of all finite subsets of $\mathcal{C} \times \mathcal{M}'$ with the well-order of Lemma 3.3, and define the support of a general element

$$r = \sum_{Y \in \mathcal{C}, u \in \mathcal{M}'} \gamma_{Y,u} Y \otimes u \in \mathfrak{k} \otimes U(\mathfrak{g})\mathfrak{g}$$

by

$$\operatorname{supp}(r) := \{(Y, u) \in \mathcal{C} \times \mathcal{M}' \mid \gamma_{Y,u} \neq 0\}.$$

Denote by $\pi : \mathfrak{k} \otimes U(\mathfrak{g})\mathfrak{g} \to U(\mathfrak{g})$ the multiplication map. We proceed by induction on $\operatorname{supp}(r)$ to show that $\pi(r) \in \mathfrak{k}$ implies $\pi(r) \in [\mathfrak{k}, \mathfrak{k}]$. Therefore, let $r \neq 0$ as above be such that $\pi(r) \in \mathfrak{k}$, and assume that for all $s \in \mathfrak{k} \otimes U(\mathfrak{g})\mathfrak{g}$ with $\operatorname{supp}(s) < \operatorname{supp}(r)$ we have $\pi(s) \in \mathfrak{k} \Rightarrow \pi(s) \in [\mathfrak{k}, \mathfrak{k}]$. By the PBW-theorem there exists a pair $(Y, u) \in \operatorname{supp}(r)$ such that u starts with $X < Y$; note that this inequality implies $X \in \mathfrak{k}$ by construction. Writing $u = Xu'$ for some PBW-monomial u' of degree $\deg(u) - 1$, we find

$$\pi(r) = \pi(r - \gamma_{Y,u} Y \otimes Xu' + \gamma_{Y,u} X \otimes Yu' + \gamma_{Y,u}[Y, X] \otimes u').$$

Now either $u' = 1$ or $u' \in \mathcal{M}'$. In the latter case, the argument s of π on the right-hand side is an element of $\mathfrak{k} \otimes U(\mathfrak{g})\mathfrak{g}$, and its support is smaller than $\operatorname{supp}(r)$, so that the induction hypothesis applies. If $u' = 1$, then the last term of $\gamma_{Y,u}[Y, X] \otimes u'$ of s is mapped into $[\mathfrak{k}, \mathfrak{k}]$ by π, and the induction hypothesis applies to the remainder of s. In either case, we find that $\pi(r) \in [\mathfrak{k}, \mathfrak{k}]$. □

This concludes the proof of Theorem 3.1. Theorem 3.2 is an easy consequence of the above proof, the proof of Theorem 2.1, and the Realization Theorem (13). As an analogue to Proposition 2.1 we have the following proposition.

PROPOSITION 3.1: *Let* $\mathfrak{g}, \mathfrak{k}, \mathfrak{m}, \mathbf{Y}, \sigma$ *and* $\mathbf{Y}_\sigma$ *be as in Proposition 2.1. Then* $\phi_{\mathbf{Y}_\sigma,\eta}$ *is a polynomial realization for any choice of* $\eta \in (\mathfrak{k}/[\mathfrak{k},\mathfrak{k}])^*$. *Indeed, if* $\mathrm{ad}(\mathfrak{m})^d X = 0$, *then the coefficients of* $\phi_{\mathbf{Y}_\sigma,\eta}(X)$ *contain only monomials of total degree* $< d$.

Proof: The proof of (5, Theorem 1.7) shows that, for given $X \in \mathfrak{g}$, not only $\chi_i(\mathbf{Y}_\sigma^{\mathbf{m}} X) = 0$ for $|\mathbf{m}| := \sum_{i=1}^n m_i$ sufficiently large, but also $\pi_{\mathfrak{k}}(\mathbf{Y}_\sigma^{\mathbf{m}} X) = 0$ for $|\mathbf{m}|$ sufficiently large. □

In the notation for parabolic subalgebras of split semisimple Lie algebras at the end of Section 2, we have $\mathfrak{p}_- = \tilde{\mathfrak{h}} \oplus [\mathfrak{p}_-, \mathfrak{p}_-]$, where $\tilde{\mathfrak{h}}$ is the subalgebra of $\mathfrak{h}$ spanned by the Chevalley generators H_α for $\alpha \in \Pi \setminus \Pi_0$. This shows that we may identify $(\mathfrak{p}_-/[\mathfrak{p}_-, \mathfrak{p}_-])^*$ with $\tilde{\mathfrak{h}}^*$, as we will do in the examples in the following section.

4. Implementation and Examples

Using de Graaf's algorithms for Lie algebras (9; 10), we implemented a function `Blattner` taking as input a finite-(say, l-)dimensional Lie algebra $\mathfrak{g}$, an ordered basis $X_1, \ldots, X_l$ of $\mathfrak{g}$, the codimension n of a subalgebra $\mathfrak{k}$ which must be spanned by the X_i with $i \leq l - n$, an element η of $\mathfrak{k}^*$ represented by its values on those X_i, and the degree d up to which the coefficients of the realization must be computed. The program does not check whether the X_i with $i \leq k$ do indeed span a basis of a subalgebra of $\mathfrak{g}$, or whether the element η really vanishes on $[\mathfrak{k}, \mathfrak{k}]$ as it should by Lemmas 3.1 and 3.5 to define a cocycle of $\mathfrak{g}$ with values in A; but if these preconditions are fulfilled, then `Blattner` returns a pair whose first component is the list of values of $\phi_{(X_{l-n+1},\ldots,X_l),\eta}$ on $X_1, \ldots, X_l$, truncated at degree d and regarded as elements of the Weyl algebra W generated by the x_i and the ∂_i, which is the second component of the output of `Blattner`. By means of some `GAP`-session printouts we show how effective Theorem 3.2 really is. Let us start with an easy example.

Example: Let $\mathfrak{g}$ be $\mathfrak{sl}_2$, denote by E, H, F the usual Chevalley basis of $\mathfrak{g}$, and let $\mathfrak{b}$ be the Borel subalgebra spanned by H and F. Identifying an element of $(\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$ with its value on H, we obtain a one-parameter family $(\phi_{E,\lambda})_{\lambda \in K}$ of polynomial representations of $(\mathfrak{g}, \mathfrak{b})$. Consider the following `GAP`-session for $\lambda = 5$.

```
gap> g:=SimpleLieAlgebra("A",1,Rationals);;
gap> B:=Basis(g,Basis(g){[2,3,1]});;
gap> L:=Blattner(g,B,1,[0,5],2);
[ [ [(5)*x_1+(-1)*x_1^2*D_1],
```

```
  [(5)*<identity ...>+(-2)*x_1*D_1],
  [(1)*D_1] ],
  <algebra-with-one of dimension infinity over Rationals> ]
```

The default basis of $\mathfrak{sl}_2$ in GAP is (E, F, H); the second input line reorders this to (F, H, E), so that the first two span $\mathfrak{b}$. By Proposition 3.1 the coefficients of $\phi_{E,\lambda}$ only contain monomials of degree ≤ 2, which justifies the last parameter to Blattner in the third input line. Note that we find precisely the realization of the introduction.

If, as in the above example, the output of Blattner is really a polynomial realization, then one can use that output in further computations. For example, we implemented a function ClosureUnderMult, whose first two arguments are lists L and M of elements of the third argument W, the Weyl algebra constructed above; and whose last argument is a function determining the multiplication to be used. Here one can think of the following multiplications: the 'normal' multiplication in W, or the commutator in W, or the action of elements of W on polynomials in the x_i, implemented under the name WeylAction. The function ClosureUnderMult computes the smallest subspace of W containing the elements of M and closed under multiplication from the left with the elements of L—provided that this space is finite-dimensional, which the program does not check.

Example (continued): The next few lines show that 1 generates a 5-dimensional $\mathfrak{sl}_2$-submodule of $K[x]$.

```
gap> M:=ClosureUnderMult(L[1],[One(L[2])],L[2],
>  function(d,p) return(WeylAction(d,p,L[2])); end);;
gap> Basis(M);
Basis( <vector space of dimension 6 over Rationals>,
[ [(1)*<identity ...>], [(1)*x_1], [(1)*x_1^2], [(1)*x_1^3],
  [(1)*x_1^4], [(1)*x_1^5] ] )
```

A somewhat more complicated example is the following.

Example: Let $\mathfrak{g}$ be the split simple Lie algebra of type G_2, let $\mathfrak{h}$ be a split Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The following GAP-session computes a realization of $(\mathfrak{g}, \mathfrak{b})$ with cocycle corresponding to the element of $\mathfrak{h}^*$ which has values $1, 1$ on the Chevalley basis of $\mathfrak{h}$.

```
gap> g:=SimpleLieAlgebra("G",2,Rationals);;
gap> B:=Basis(g,Basis(g){Concatenation([7..14],[1..6])});;
gap> L:=Blattner(g,B,6,[0,0,0,0,0,0,1,1],6);;
```

Here 6 is the nilpotence degree of the action of $\mathfrak{u}_+$ on $\mathfrak{g}$. Including the complete realization here would not be very instructive. Instead, we check that the first component of L does indeed spans a Lie algebra:

```
gap> ClosureUnderMult(L[1],L[1],L[2],Lie);
<vector space over Rationals, with 14 generators>
```

Finally, we construct the irreducible module of highest weight $(1,1)$ as in the third example of Section 4; see Section 5 for an explanation of why these finite-dimensional irreducible modules emerge in this way.

```
gap> M:=ClosureUnderMult(L[1],[One(L[2])],L[2],
function(d,p) return(WeylAction(d,p,L[2])); end);
<vector space over Rationals, with 64 generators>
```

Now consider an example where the isotropy subalgebra is parabolic, but not a Borel subalgebra.

Example: Let $\mathfrak{g}$ be $\mathfrak{sl}_3$, and let $\mathfrak{p}_-$ be the parabolic subalgebra corresponding to $\Pi_0 = \{\alpha_2\}$ in the notation of page 5 and using the standard labelling of (4). Then $\mathfrak{p}_- = [\mathfrak{p}_-, \mathfrak{p}_-] \oplus KH_1$, where H_1 is the element in the Chevalley basis of $\mathfrak{h}$ corresponding to α_1. The following GAP-session computes a realization of $(\mathfrak{g}, \mathfrak{p}_-)$ with cocycle whose value on H_1 is 2.

```
gap> g:=SimpleLieAlgebra("A",2,Rationals);;
gap> B:=Basis(g,Basis(g){[2,4,5,6,7,8,1,3]});;
gap> L:=Blattner(g,B,2,[0,0,0,0,2,0],2);
[ [ [(-1)*x_1*D_2], [(2)*x_1+(-1)*x_1*x_2*D_2+(-1)*x_1^2*D_1],
    [(-1)*x_2*D_1], [(-1)*x_1*x_2*D_1+(2)*x_2+(-1)*x_2^2*D_2],
    [(2)*<identity ...>+(-2)*x_1*D_1+(-1)*x_2*D_2],
    [(1)*x_1*D_1+(-1)*x_2*D_2], [(1)*D_1], [(1)*D_2] ],
  <algebra-with-one of dimension infinity over Rationals> ]
gap> ClosureUnderMult(L[1],L[1],L[2],Lie);
<vector space over Rationals, with 8 generators>
```

This shows that `L[2]` is indeed the basis of an 8-dimensional Lie algebra. Leaving out the zero order terms we retrieve one of the primitive Lie algebra's from Lie's classification (15).

```
gap> M:=ClosureUnderMult(L[1],[One(L[2])],L[2],
> function(d,p) return(WeylAction(d,p,L[2])); end);;
gap> List(Basis(M));
[ [(1)*<identity ...>], [(1)*x_1], [(1)*x_2], [(1)*x_1^2],
  [(1)*x_1*x_2], [(1)*x_2^2] ]
```

The following example shows that the applications of our Realization Formula are not restricted to semisimple $\mathfrak{g}$.

Example: Let $\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is an Abelian two-dimensional ideal and irreducible as an $\mathfrak{sl}_2$-module, and let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_2$ spanned by

H and F. Then $\mathfrak{n}+KE$ is a subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{b}$; let X_{-1}, X_1 be a basis of $\mathfrak{n}$ satisfying $[H, X_i] = iX_i,\ i = -1, 1$. The following GAP-session computes the polynomial realization $\phi_{(E,X_1,X_{-1}),\eta}$ of the pair $(\mathfrak{g}, \mathfrak{b})$, where $\eta \in (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$ is determined by its value 5 on H. The printout below starts directly after the construction of $\mathfrak{g}$ with ordered basis $B = (H, F, E, X_1, X_{-1})$.

```
gap> L:=Blattner(g,B,3,[5,0],2);
[ [ [(5)*<identity ...>+(-2)*x_1*D_1+(-1)*x_2*D_2+(1)*x_3*D_3],
  [(5)*x_1+(-1)*x_1^2*D_1+(-1)*x_2*D_3],
  [(-1)*x_3*D_2+(1)*D_1],
  [(1)*D_2],
  [(1)*D_3] ],
  <algebra-with-one of dimension infinity over Rationals> ]
```

5. Conclusion

As Richter explains in (18), it is no coincidence that 1 generates a finite-dimensional module in the first two examples of Section 4; this is a consequence of the celebrated Borel-Weil theorem (3). Indeed, suppose that K is an algebraically closed field of characteristic 0, let G be a connected reductive affine algebraic group over K, and let B_- be a Borel subgroup of G. To any algebraic group homomorphism λ from B_- into the multiplicative group of K we associate the algebraic line bundle $L_\lambda := G \times_{B_-} Kv_\lambda$ over G/B_-; here v_λ spans the 1-dimensional B_--module corresponding to λ. The group G acts on L_λ in a natural way, hence also on the space V_λ of regular sections of L_λ. The Borel-Weil theorem states that if λ is dominant, then V_λ is a finite-dimensional irreducible module of highest weight λ.

Suppose that λ is indeed dominant, and let $s_0 : G/B_- \to L_\lambda$ be a highest weight vector in V_λ. Let N_+ be the unipotent radical of the Borel subgroup opposite to B_-. Then N_+ is isomorphic to an affine space, and the map $\iota : n \mapsto nB_-$ is an open dense embedding of N_+ into G/B_- (2). We claim that s_0 does not vanish on $\iota(N_+)$. Indeed, as s_0 is a highest weight vector, we have $ns_0 = s_0$ for all $n \in N_+$, or, equivalently,

$$ns_0(n^{-1}gB_-) = s_0(gB_-) \text{ for all } n \in N_+,\ g \in G.$$

As s_0 is non-zero, it cannot vanish on all of the open dense subset $\iota(N_+)$ of G/B_-; hence there exists an $n_0 \in N_+$ such that $s_0(n_0B_-) \neq 0$. Replacing g by nn_0 in the equation above, we find $ns_0(n_0B_-) = s_0(nn_0B_-)$. The left-hand side is non-zero by choice of n_0, hence so is the right-hand side. We conclude that s_0 indeed does not vanish on N_+.

From this it follows that if s is any section of L_λ, then its restriction to $\iota(N_+)$ can be written as fs_0 for some regular function f on the affine space N_+. The representation of G on sections translates into a representation of G on these coefficients f, and the associated action of $\mathfrak{g}$ on those functions is given by the first order differential operators computed in Section 4. This explains why 1, which corresponds to the section s_0, is the highest weight vector of a finite-dimensional irreducible module of highest weight λ. A similar argument should hold in the case where B_- is replaced by a larger parabolic subgroup, explaining the finite-dimensional module constructed in Example 4.4.

On the whole, it is fair to say that the objects constructed in the paper at hand have already been known abstractly for a long time. Our contribution is their explicit and automatic computation in a general setting. Our GAP-program—a copy of which can be obtained by sending an e-mail to the author—has proved a useful tool in the construction of Lie algebras of differential operators, and could very well be useful in the construction of new quasi-exactly solvable Hamiltonians.

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