COUNTING COMPONENTS OF THE NULL-CONE ON TUPLES

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Abstract

For a finite-dimensional representation $\rho : G \to \mathrm{GL}(M)$ of a group G the diagonal action of G on M^p, p-tuples of elements of M, is usually poorly understood. The algorithm presented here computes a geometric characteristic of this action in the case where G is connected and reductive, and ρ is a morphism of algebraic groups: The algorithm takes as input the weight system of M, and it returns the number of irreducible components $c(M^p)$ of the *null-cone* of G on M^p for large p. The paper concludes with a theorem that if the characteristic is zero and if G is semisimple, then only few M have the property that $c(M^p)$ is small for all p.

1. Introduction

Though for many classical geometric objects 'normal forms' under the action of a group G are known—think of complex $n \times n$-matrices under conjugation or bilinear forms on $\mathbb{C}^n$ under the natural action of GL_n—very little is usually known about the *diagonal action* of that group on p-tuples of those objects. This paper explains how to compute an interesting geometric characteristic of the group action on p-tuples for large p: the number of components of the null-cone.

More specifically, let K be an algebraically closed field of characteristic 0 and let G be a group. Let M be a finite-dimensional vector space over K and $\rho : G \to \mathrm{GL}(M)$ a representation of G. Then an element of M is called *nilpotent* if it cannot be distinguished from 0 by G-invariant polynomials on M.

Example 1.1.
(1) If $G := \mathrm{SL}_n$ acts by conjugation on $M := M_n$, the space of $n \times n$-matrices, then the algebra of invariant polynomials is generated by the coefficients of the characteristic polynomial of an element of M_n, so that the nilpotent elements of M_n are precisely the nilpotent matrices.
(2) If $G := \mathrm{SL}_n \times \mathrm{SL}_m$ acts by conjugation on $M := M_{n,m}$, then invariant polynomials are constant if $n \neq m$ and polynomials in the determinant if $n = m$. In the first case, all matrices are nilpotent elements, and in the second case, a matrix is nilpotent if and only if it is singular.

The set of nilpotent elements form a G-stable, closed cone in M, called the *null-cone* in M and denoted $\mathcal{N}_G(M)$ or $\mathcal{N}(G)$ if G is clear from the context. We denote the number of irreducible components of $\mathcal{N}(M)$ by $c_G(M) = c(M)$. We are interested in the null-cone $\mathcal{N}(M^p)$ in M^p, the direct sum of p copies of M, on which G acts diagonally. It can be shown that $c(M^p)$ is an ascending function of p which stabilises at some finite p_0; that is: for all $p \geq p_0$ the null-cone $\mathcal{N}(M^p)$ has the same number of irreducible components as $\mathcal{N}(M^{p_0})$. This phenomenon was first observed

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by Kraft and Wallach [8] in the case of reductive group actions to be treated soon; that it occurs in general is proved in [4]. The present paper presents an algorithm to compute the limit $\lim_{p\to\infty} c(M^p)$ modules M over connected, reductive groups.

Example 1.2. (1) The null-cone of SL_n by simultaneous conjugation on M_n^p is always irreducible and its elements are precisely those p-tuples of nilpotent matrices that can be simultaneously conjugated into upper triangular form; it follows readily that $c(M^p) = 1$ for all p. Representations M for which $\lim_{p\to\infty} c(M^p)$ is small are rare; see Section 4.

(2) The null-cone of $\mathrm{SL}_n \times \mathrm{SL}_m$ on $M_{n,m}^p$, where we may assume that $n \geq m$, is all of $M_{n,m}^p$ for $p < n/m$, while it has exactly m irreducible components for $p > \lceil n/m \rceil$, given by

$$C_k := \{(A_1, \ldots, A_p) \mid \text{ there exists a } k\text{-dimensional subspace } U \text{ of } K^m \text{ for which } \dim \sum_i (A_i U) < k\frac{n}{m}\},\ k = 1, \ldots, m;$$

hence $\lim_{p\to\infty} c(M^p) = m$. This fact, as well as what happens for $p = \lceil n/m \rceil$, is explained in [4].

The number m of irreducible copies in the second example was first found—for concrete values m and n—using the algorithm that I want to explain here. Section 2 recalls the results of Kraft and Wallach at the heart of the algorithm. In Section 3 we see how these results lead to the problem of counting open cells in the complement of a real hyperplane arrangement in a Weyl chamber. I give a short proof of a version of Zaslavsky's formula for this number of cells [14]—and the same proof works, in fact, for Zaslavsky's original formula—and use it in said algorithm computing $\lim_{p\to\infty} c(M^p)$. Finally, in Section 4 we apply our insights to the problem of describing all M for which $\lim_{p\to\infty} c(M^p)$ is small; we saw one example of such a module above.

2. Components of the null-cone on M^p

We fix a connected, reductive algebraic group G over an algebraically closed field K, and a finite-dimensional rational G-module M—for general theory of algebraic groups and their representations, I refer to [2, 13]. In the examples given in the Introduction, G and M are of this type. In this setting, there is a beautiful geometric tool for describing the null-cone $\mathcal{N}(M)$: If there exists for $v \in M$ a one-parameter subgroup (1-PSG) $\lambda : K^* \to G$ such that $\lim_{t\to 0} \lambda(t)v$ is zero—by which we simply mean that $\lambda(t)v$ is of the form $\sum_{i=1}^d t^i v_i$ with $v_i \in M$—then clearly v is nilpotent. The Hilbert-Mumford criterion [10, 12] states that the converse is also true, that is: for every nilpotent vector $v \in M$ there exists a 1-PSG λ such that $\lim_{t\to 0} \lambda(t)v = 0$.

This remarkable characterisation of the null-cone of a reductive group representation found an important application in the notion of 'optimal 1-PSGs', that is: those that steer a given nilpotent vector to zero in the most efficient way; see [12] and the references there. Nilpotent vectors having equivalent optimal 1-PSGs form a stratum of a stratification of the null-cone by smooth, irreducible locally closed subvarieties [7]. This stratification can be computed with an algorithm given in [11]. In the present paper, however, we concentrate on more modest geometric information, but for null-cones in large representations: we want to determine $\lim_{p\to\infty} c(M^p)$, the number of irreducible components of M^p for large p.

Here is Kraft and Wallach's set-up for this problem: Let T be a maximal torus in G, denote by $X = X(T)$ the group of characters $T \to K^*$, and denote by $Y = Y(T)$ the group of 1-PSGs $K^* \to T$. Then X and Y are finitely generated free abelian groups, written additively, dual to each other relative to the natural pairing $\langle .,. \rangle : X \times Y \to X(K^*) = \mathbb{Z}$ given by composition; here $X(K^*)$ is identified with $\mathbb{Z}$ by mapping the generator $s \mapsto s$ of $X(K^*)$ to $1 \in \mathbb{Z}$.

Example 2.1. In SL_n we may choose T to consist of the unimodular diagonal matrices. Then the 1-PSGs $\check{\alpha}_i : K^* \to T,\ i = 1, \ldots, n-1$ sending $s \in K^*$ to the diagonal matrix $\mathrm{diag}(1, \ldots, 1, s, s^{-1}, 1, \ldots)$ form a basis of Y, and the characters $\omega_i : T \to K^*,\ i = 1, \ldots, n-1$ mapping t to $t_{1,1} \cdots t_{i,i}$ form the dual basis of X.

For $\chi \in X$ we let $M_\chi := \{v \in M \mid tv = \chi(t)v \text{ for all } t \in T\}$ be the weight space of χ in M; M is the direct sum of these as χ runs through the weight set

$$X_M := \{\chi \in X \mid M_\chi \neq 0\}.$$

For $\lambda \in Y$ we set

$$M(\lambda) := \{v \in M \mid \lim_{s \to 0} \lambda(s)v = 0\} \text{ and}$$
$$X_M(\lambda) := \{\chi \in X_M \mid \langle \chi, \lambda \rangle > 0\};$$

a set of the latter form is called a *half* of X_M. Note that $M(\lambda)$ is the sum of all M_χ with $\chi \in X_M(\lambda)$. By the Hilbert-Mumford criterion and the conjugacy of maximal tori in G, the null-cone $\mathcal{N}(M)$ is the union of all sets of the form

$$C(\lambda) := GM(\lambda)$$

as λ runs through Y. As $C(\lambda)$ is the image under $G \times M(\lambda)$ under the action, it is an irreducible set. Moreover, from the fact that $M(\lambda)$ is stable under the parabolic subgroup

$$P(\lambda) := \{g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$$

it follows that $C(\lambda)$ is closed. We have thus covered $\mathcal{N}(M)$ by closed, irreducible subsets, so the irreducible components of the null-cone are among them. However, there are still inclusions among the $C(\lambda)$; having four different causes:

(1) A half $X_M(\lambda)$ may be equal to $X_M(\lambda')$; this is equivalent to the statement that $\langle \chi, \lambda \rangle$ has the same sign as $\langle \chi, \lambda' \rangle$ for all $\chi \in X_M$. In this case we have $C(\lambda) = C(\lambda')$.
(2) A half $X_M(\lambda)$ may be strictly contained in $X_M(\lambda')$ for some other λ'; then clearly $C(\lambda) \subseteq C(\lambda')$. We call $X_M(\lambda)$ a *maximal half* if it is not strictly contained in any other half.
(3) Suppose that $X_M(\lambda)$ and $X_M(\lambda')$ are distinct maximal halfs, but that they are conjugate by an element of the Weyl group $W := N_G(T)/T$. Then the action of G 'smears out' $M(\lambda)$ and $M(\lambda')$ to the same set $C(\lambda) = C(\lambda')$.
(4) Finally, even if $X_M(\lambda)$ and $X_M(\lambda')$ are maximal halfs that are not conjugate under the Weyl group, $C(\lambda)$ may be equal to, or strictly contained in $C(\lambda')$—for instance, G may have a dense orbit in M, in which case some of the $C(\lambda)$ are all of M.

Kraft and Wallach showed that the last phenomenon does not occur if we pass from M to a direct sum of sufficiently many copies. More precisely, let p be a natural number and consider the representation of G on the direct sum M^p of

$p \geq 1$ copies of M. Note that $X_{M^p} = X_M$ and therefore $X_{M^p}(\lambda) = X_M(\lambda)$; we will continue to write X_M and $X_M(\lambda)$, and we write $C^p(\lambda)$ for $C(\lambda)$ in M^p.

Proposition 2.2 (Kraft and Wallach [8]). *For p sufficiently large the $C^p(\lambda)$ for which $X_M(\lambda)$ are maximal halfs are precisely the irreducible components of the null-cone $\mathcal{N}(M^p)$, and $C^p(\lambda) = C^p(\lambda')$ if and only if $X_M(\lambda)$ and $X_M(\lambda')$ are conjugate under W.*

For completeness, I include a version of Kraft and Wallach's proof.

Proof. Assume that $C^p(\lambda')$ contains $C^p(\lambda)$. Choose an $m = (v_1, \ldots, v_p)$ in $M^p(\lambda)$ such that the v_i span $M(\lambda)$; this is possible if p is large enough. By assumption there exists a $g \in G$ such that $gm \in M^p(\lambda')$, and as the v_i span $M(\lambda)$ we then have $gM(\lambda) \subseteq M(\lambda')$. As we noted before, $M(\lambda)$ is stable under the parabolic subgroup $P = P(\lambda)$ containing T, and similarly $M(\lambda')$ is stable under $P' = P(\lambda')$, which also contains T. A consequence of the Bruhat decomposition is that $G = \bigcup_{w \in W} P'\tilde{w}P$ for any two parabolic subgroups P and P' containing T, hence we may write $g = p'\tilde{w}p$ for some $p' \in P', w \in W, p \in P$. But then we find $\tilde{w}M(\lambda) \subseteq M(\lambda')$, so that $X_M(\lambda)$ and $X_M(\lambda')$ are W-conjugate. □

In fact, Kraft and Wallach prove something slightly more general, and give an explicit upper bound for the smallest p with this property. This is only the starting point of their paper; they proceed to exploit these observations in many concrete examples and special cases such as θ-representations. In the present paper we use the following direct consequence of their proposition: *the number $\lim_{p\to\infty} c(M^p)$ of irreducible components of $\mathcal{N}(M^p)$ for large p is equal to the number of W-orbits on maximal halfs.*

3. Counting components and faces

Retaining the notation $G, T, X = X(T), Y = Y(T), M, X_M, X_M(\lambda)$ of the previous section, we now face the problem of counting W-orbits on maximal halfs $X_M(\lambda)$. For the sake of geometric intuition it is convenient to work with $X_{\mathbb{R}} := X \otimes_{\mathbb{Z}} \mathbb{R}$ and $Y_{\mathbb{R}} := Y \otimes_{\mathbb{Z}} \mathbb{R}$, whose elements we call *virtual characters* and *virtual 1-PSGs*, respectively. The pairing $\langle .,. \rangle$ extends to a non-degenerate $\mathbb{R}$-bilinear pairing $X_{\mathbb{R}} \times Y_{\mathbb{R}} \to \mathbb{R}$, and we extend our notation $X_M(\lambda)$ to virtual 1-PSGs $\lambda \in Y_{\mathbb{R}}$.

Let λ be a non-zero virtual 1-PSG. By definition, $X_M(\lambda)$ is the intersection of X_M with the open half-space in $X_{\mathbb{R}}$ where λ is positive. Let S be the set of weights in X_M where λ vanishes. Then any virtual 1-PSG λ' lying close enough to λ in the subspace $\{\mu \in Y_{\mathbb{R}} \mid \langle S, \mu \rangle = 0\}$ of $Y_{\mathbb{R}}$ will define the same half as λ, and as said subspace is defined by rational equations, it follows that $X_M(\lambda) = X_M(\lambda')$ for some non-virtual $\lambda' \in Y$. In other words, we do not introduce any new halfs by varying λ over $Y_{\mathbb{R}}$ rather than over Y.

Next note that a $\lambda \in Y_{\mathbb{R}}$ vanishing on some non-zero weight γ of M will never define a maximal half: by perturbing λ slightly, we 'gain' γ to the positive half-space defined by λ, without losing any weights of M already lying there. We conclude that, to count the maximal halfs of X_M, we may restrict our attention to the virtual 1-PSGs lying in

$$Z := Y_{\mathbb{R}} \setminus \bigcup_{\gamma \in X_M \setminus \{0\}} \gamma^{\perp},$$

where $\gamma^\perp$ denotes the annihilator in $Y_\mathbb{R}$ of γ. Now one has $X_M(\lambda) = X_M(\lambda')$ if and only if λ and λ' lie in the same connected component of Z, so if F is such a component, then we may write $X_M(F)$ for $X_M(\lambda), \lambda \in F$. In accordance, we call the *component* F maximal if $X_M(F)$ is a maximal half of X_M. We are thus led to count *W-orbits on maximal components of Z.*

If $X_M = -X_M$, then *all* components of Z are maximal, as then every virtual 1-PSG $\lambda \in Z$ has exactly half of the non-negative weights in M in its positive halfspace. This is the case, for instance, if all irreducible subquotients of M are self-dual.

It is the problem of counting W-orbits on *all* components of Z, rather than just those on *maximal* components, that we first deal with, regardless of whether $X_M = -X_M$. To this end, choose a system $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq X$ of simple roots in the root system of (G, T), let $P_+ \subseteq Y_\mathbb{R}$ be the (dual) Weyl chamber defined by

$$P_+ := \{\lambda \in Y_\mathbb{R} \mid \langle \alpha_i, \lambda \rangle \geq 0 \text{ for all } i\},$$

and let P_+° be the interior of P_+. Recall that if G is semisimple, then the α_i form a basis of $X_\mathbb{R}$, and the closed, convex cone P_+ contains non non-zero subspaces of $Y_\mathbb{R}$. In general we write T^1 for the maximal torus in T of the derived subgroup (G, G), and T^2 for the connected component of the centre of G. Then the multiplication map $T^1 \times T^2 \to T$ is surjective and has a finite kernel, so that the corresponding map $Y(T_1) \times Y(T_2) \to Y(T)$ is an injective map between free Abelian groups of the same rank. Tensoring with $\mathbb{R}$ yields a linear isomorphism $Y_\mathbb{R} \cong Y^1_\mathbb{R} \oplus Y^2_\mathbb{R}$, where $Y^i = Y(T_i)$. Similarly, the injective pull-back $X(T) \to X(T_1) \times X(T_2)$ yields a linear isomorphism $X_\mathbb{R} \cong X^1_\mathbb{R} \oplus X^2_\mathbb{R}$, where $X^i = X(T_i)$. Under this identification $X^i_\mathbb{R}$ is the annihilator of $Y^{2-i}_\mathbb{R}$ for $i = 1, 2$, and $X^1_\mathbb{R}$ is spanned by the simple roots. We find that in this case the Weyl chamber equals

$$P_+ = ((P_+) \cap Y^1_\mathbb{R}) \times Y^2_\mathbb{R},$$

where the first factor is a cone not containing non-zero subspaces of $Y^1_\mathbb{R}$, as before. The following lemma shows that we need only count the components of Z that intersect P_+.

Lemma 3.1. *Every W-orbit on components of Z contains a unique component that intersects P_+. This component intersects P_+° in a connected component of $P_+^\circ \cap Z$.*

Proof. It is well known that P_+ is a fundamental domain of the W-action on $Y_\mathbb{R}$, and that the unique W-invariant map $\phi : Y_\mathbb{R} \to P_+$ extending the identity on P_+ is a topological quotient map by W. This readily implies that every W-orbit on connected components of Z contains an element intersecting P_+. Now let F be such a component intersecting P_+. Then $\phi(F)$, being the image of a connected set under a continous map, is connected, and by the W-stability of X_M it is contained in $Z \cap P_+$. Moreover, $\phi(F)$ contains $\phi(F \cap P_+) = F \cap P_+$, which is a connected component of $Z \cap P_+$, and together with the connectedness of $\phi(F)$ this implies $\phi(F) = F \cap P_+$. It follows that if $w \in W$ and wF intersects P_+, as well, then $wF \cap P_+ = \phi(wF) = \phi(F) = F \cap P_+$, so F and wF are not disjoint and hence equal. The last statement is obvious. □

Thus counting the W-orbits on components of Z boils down to counting the connected components of

$$Z_+ := Z \cap P_+^\circ.$$

It is good to describe this set in more detail: one obtains Z_+ by leaving out from the closed convex cone P_+ all hyperplanes bounding it, as well as all hyperplanes of the shape $\gamma^\perp$ with $\gamma \in X_M$. However, many of the latter hyperplanes do not intersect P_+° at all, can therefore be ignored in the construction of Z_+. We call a virtual weight $\gamma \in X_\mathbb{R}$ *relevant* if $\gamma \neq 0$ and if $\gamma^\perp$ does intersect P_+°.

Lemma 3.2. *A virtual weight $\gamma \in X_\mathbb{R}$ is relevant if and only if it is not of the form*

$$\gamma = \sum_i c_i \alpha_i \text{ with } \forall_i c_i \geq 0 \text{ or } \forall_i c_i \leq 0.$$

In other words, γ is relevant if and only if it does not lie in the convex cone spanned by the positive simple roots α_i, nor in the convex cone spanned by the $-\alpha_i$.

Proof. If $\gamma \neq 0$ is in the convex cone spanned by the α_i and if $\lambda \in P_+^\circ$, then $\langle \alpha_i, \lambda \rangle > 0$ for all α_i and so $\langle \gamma, \lambda \rangle > 0$; it follows that γ and $-\gamma$ are both not relevant.

On the other hand, suppose that γ is in none of the two cones described above. Write $\gamma = \gamma^1 + \gamma^2$ where $\gamma^i \in X_\mathbb{R}^i$, and write $\gamma^1 = \sum_i c_i \alpha_i$. Suppose first that $\gamma^2 \neq 0$. Choose any $\lambda^1 \in P_+^\circ \cap Y_\mathbb{R}^1$, and choose a $\lambda^2 \in Y_\mathbb{R}^2$ such that $\langle \gamma^2, \lambda^2 \rangle = -\langle \gamma^1, \lambda^1 \rangle$. We then have $\lambda = \lambda^1 + \lambda^2 \in P_+^\circ$ and $\langle \gamma, \lambda \rangle = 0$, so γ is relevant.

If $\gamma^2 = 0$, then the c_i do not all have the same sign by assumption; suppose that $c_{i_-} < 0$ and $c_{i_+} > 0$ and let (x_i) be the basis of $Y_\mathbb{R}^1$ dual to the α_i (and vanishing on $X_\mathbb{R}^2$). Let λ_1 be as before, and compute $\langle \gamma, \lambda \rangle =: c$. If $c \leq 0$, then $\lambda := \lambda_1 - \frac{c}{c_{i_+}} x_{i_+}$ annihilates γ and lies in $P_+^\circ(\cap Y_\mathbb{R}^1)$, while if $c \geq 0$ then $\lambda := \lambda_1 - \frac{c}{c_{i_-}} x_{i_-}$ does the job. In either case, γ is relevant. □

We construct the set X_M^{rel} of *relevant rays* as follows:

$$X_M^{\mathrm{rel}} := \{\mathbb{R}_+ \gamma \mid \gamma \in X_M \text{ is relevant}\};$$

for $\gamma \in X_M^{\mathrm{rel}}$ and $\lambda \in Y_\mathbb{R}$ we will use the notation $\langle \gamma, \lambda \rangle$ for the sign of $\langle \gamma', \lambda \rangle$ with γ' any representative of γ. We are now ready for our first main result. Let $\mathcal{L}$ be the collection of all sets of the form

$$S^\perp \cap P_+ \text{ where } S \subseteq X_M^{\mathrm{rel}} \cup \Delta,$$

and order $\mathcal{L}$ partially by inclusion. In other words: an element of $\mathcal{L}$ is the intersection of some hyperplanes of the form $\gamma^\perp$ with $\gamma \in X_M$ relevant and some of the hyperplanes bounding P_+, intersected with the cone P_+. The lattice has two types of elements: $V \in \mathcal{L}$ is either a linear subspace of $Y_\mathbb{R}$, or else it is a proper convex polyhedral cone in its linear span. It is easy to see that $\mathcal{L}$ is a lattice with smallest element $O := (X_M \cup \Delta)^\perp$ (note that this subspace is contained in P_+) and largest element P_+. Of course, if $X_M \cup \Delta$ spans $X_\mathbb{R}$—in particular, if G is semisimple—then O is the zero space.

From the lattice $\mathcal{L}$ we construct its set of *faces*, as follows: for $V \in \mathcal{L}$ let $\mathcal{F}(V)$ be the set of connected components of

$$V \setminus \bigcup_{W \in \mathcal{L}, W \subsetneq V} W,$$

and let $\mathcal{F}$ be the union of all $\mathcal{F}(V)$, $V \in \mathcal{L}$. Let $\#(V)$ be the cardinality of $\mathcal{F}(V)$, so that Z_+ has $\#(P_+)$ connected components. The numbers $\#(V)$ can now be computed as follows.

Theorem 3.3 (Zaslavsky type formula)**.** *For $V \in \mathcal{L}$ let $\chi(V)$ be $(-1)^{\dim V}$ if V is a linear subspace of $Y_{\mathbb{R}}$ and 0 if V is a proper polyhedral cone. Then*

$$\sum_{W \in \mathcal{L}, W \subseteq V} (-1)^{\dim W} \#(W) = \chi(V)$$

for all $V \in \mathcal{L}$.

Proof. Let V be an element of $\mathcal{L}$ and let $\pi : Y_{\mathbb{R}} \to Y_{\mathbb{R}}/O$ be the canonical projection. The set

$$\{\pi(F) \mid F \in \mathcal{F},\ F \subseteq V\} \cup \{\infty\}$$

defines a CW-complex on the one-point compactification $\pi(V) \cup \{\infty\}$ of $\pi(V)$. If V is a proper cone, then $\pi(V) \cup \{\infty\}$ is homotopic to a line segment with endpoints 0 and ∞, hence has Euler characteristic 1. If V is a linear subspace, then $\pi(V) \cup \{\infty\}$ is homeomorphic to a sphere of dimension $\dim(V) - \dim(O)$, so that it has Euler characteristic $1 + (-1)^{\dim(V) - \dim(O)}$. In either case, we find

$$\sum_{F \in \mathcal{F}, F \subseteq V} (-1)^{\dim F} = \chi(V),$$

where we left out $\{\infty\}$ and multiplied by $(-1)^{\dim O}$. Grouping together the faces F for which the minimal element of $\mathcal{L}$ containing F is the same, we obtain the formula of the theorem. □

This theorem allows for the following algorithm for computing the number of W-orbits on (not necessarily maximal) components of Z:

(1) Construct the set X_M^{rel} of relevant rays;
(2) Construct the lattice $\mathcal{L}$;
(3) Let $\zeta : \mathcal{L} \times \mathcal{L} \to \{-1, 0, 1\}$ be the matrix the matrix whose rows and columns are labelled by the elements of $\mathcal{L}$ with entries $\zeta(V, W) = (-1)^{\dim W}$ if W is contained in V and 0 otherwise;
(4) Let $\chi : \mathcal{L} \to \{-1, 0, 1\}$ be the column vector defined earlier;
(5) Then # is the solution of the linear system $\zeta \cdot \# = \beta$.

As noted before, this algorithm counts the number $\lim_{p \to \infty} c(M^p)$ in case $X_M^{\mathrm{rel}} = -X_M^{\mathrm{rel}}$, while otherwise we still have to select the *maximal* components among the $\#(P_+)$ components thus found.

If $G = T$, then the Weyl group is trivial and we recover Zaslavsky's formula [14] for the number of cells in the complement of a hyperplane arrangement. It is not hard to adapt the proof above to the case where the hyperplanes are not required to pass through 0, so this gives an efficient proof of his formula.

Example 3.4. Suppose that $G = \mathrm{GL}_2$ and $M = K^2$. Choose for T the torus of diagonal matrices; then T^1 is the maximal torus in SL_2 consisting of diagonal matrices with determinant 1 and T^2 consists of multiples of the identity matrix. Note that $T^1 \cap T^2 = \{\pm I\}$. As a basis for $Y_{\mathbb{R}}^1$ we choose the 1-PSG $\lambda_1 : t \mapsto \mathrm{diag}(t, t^{-1})$ and as a basis for $Y_{\mathbb{R}}^2$ we choose the 1-PSG $\lambda_2 : t \mapsto \mathrm{diag}(t, t)$. As a basis for $X_{\mathbb{R}}^1$ we choose the weight $\gamma_1 : \mathrm{diag}(t, t^{-1}) \mapsto t$ and for $X_{\mathbb{R}}^2$ we choose the weight $\gamma_2 : \mathrm{diag}(t, t) \mapsto t$.

On the other hand, the two characters $\chi_i : \mathrm{diag}(t_1, t_2) \mapsto t_i$ form a basis of X. The pull-back $X \to X^1 \times X^2$ maps $\chi_1 - \chi_2$ to $2(\gamma_1, 0)$ and $\chi_1 + \chi_2$ to $2(0, \gamma_2)$. This allows us to view γ_1, γ_2 as the basis $(\chi_1 - \chi_2)/2, (\chi_1 + \chi_2)/2$ of $X_{\mathbb{R}}$, which is dual to the basis of $Y_{\mathbb{R}}$ given by λ_1, λ_2.

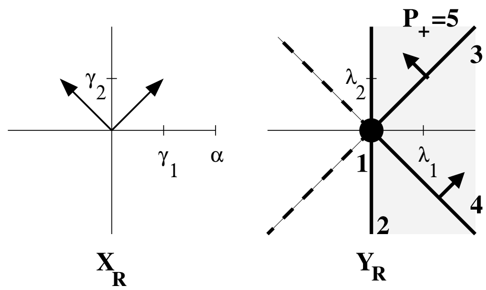

FIGURE 1. Weight system (left) and subspace lattice (right).

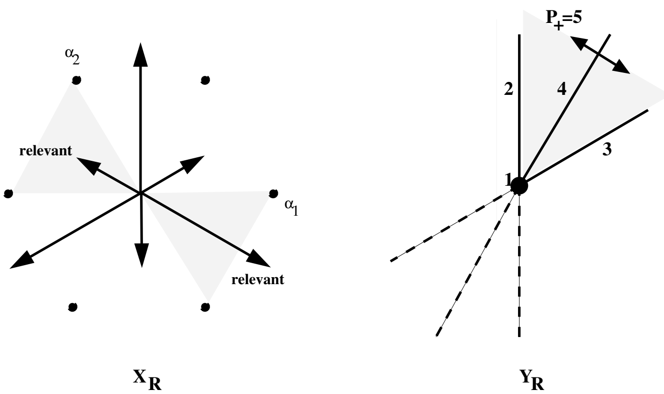

FIGURE 2. Weight system (left) and subspace lattice (right).

The weight vectors of T in M are the standard basis vectors e_1, e_2, with weights $\chi_1 = \gamma_1 + \gamma_2$ and $\chi_2 = -\gamma_1 + \gamma_2$, respectively. In Figure 1 these weights in $X_{\mathbb{R}}$ are drawn on the left, and the hyperplanes (lines) in $Y_{\mathbb{R}}$ annihilating them are drawn on the right, together with the closed half plane P_+. The lattice $\mathcal{F}$ has 5 elements, depicted in the figure. The minimal element is the origin, the λ_2-axis is a linear subspace of $Y_{\mathbb{R}}$, and the other elements are proper cones. Thus the linear system to be solved is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \#_1 \\ \#_2 \\ \#_3 \\ \#_4 \\ \#_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and we find $\# = (1, 2, 1, 1, 3)^t$ as is obvious from the picture. Note however, that only one of the 3 components of Z_+ is maximal, so the null-cone in any M^p is irreducible. In fact, it is clearly the whole space, as we have $\lim_{t \mapsto 0}(tI)(v_1, \ldots, v_p) = 0$ for all $v_1, \ldots, v_p \in M$.

Example 3.5. Let $G = \mathrm{SL}_3$ and let M be the space of homogeneous polynomials in x_1, x_2, x_3 of degree 2. Let T be the torus of Example 2.1. Then the weight vectors in M are the monomials $x_i x_j$. Figure 2 depicts their weights and the root system on the left. Note that only two weights are relevant, and one is a negative scalar multiple of the other. The linear system $\zeta\# = \beta$ to be solved is identical to that of the previous example, exept that now *all* non-minimal elements of $\mathcal{L}$ are cones (and not subspaces), so that $\beta = (1,0,0,0,0)^t$. We find that Z_+ has two connected components (this is obvious from the picture), and both are maximal. Hence $\mathcal{N}(M^p)$ has two irreducible components for $p \gg 0$. In fact, one may describe these two components by the corresponding quadrics in $\mathbb{P}^2$, as follows: If $(q_1,\ldots,q_p)$ lies in $\mathcal{N}(M^p)$, then all q_i define singular quadrics. Assume that all are of rank 2, so that each q_i defines a pair of lines, whose intersection is the radical of q_i. The two components of $\mathcal{N}(M^p)$ are now described as follows: the first component consists of all p-tuples (q_i) for which all radicals coincide, and the second component consists of all p-tuples for which all line pairs share a line [4].

A way to compute the number of *maximal* components of Z_+ is now the following:

(1) First compute $\#(P_+)$ as above.
(2) Set $S := \emptyset$.
(3) Generate a point λ in P_+ randomly, and compute its corresponding *sign vector*: the function $s : X_M^{\mathrm{rel}} \to \{-1,0,1\}$ that assigns to $\gamma \in X_M$ the sign $\langle\gamma,\lambda\rangle$. Discard s if it contains a 0—generically, this will not be the case—and otherwise add s to S; note that s uniquely determines the connected component of Z_+ containing λ.
(4) Repeat step 3 until $|S| = \#(P_+)$. We have then found representatives for all connected components of Z_+.
(5) Discard those $s \in S$ for which there exists a $\gamma \in X_M^{\mathrm{rel}}$ such that $s(\gamma) = -1$ and the vector s' obtained by flipping $s(\gamma)$ to $+1$ while keeping the other entries unchanged is also in S.
(6) The remaining sign vectors are representatives of the maximal components of Z_+.

The last steps are justified by the following lemma.

Lemma 3.6. *If F is a component of Z_+ which is not maximal, then there exist a $\gamma \in X_M^{\mathrm{rel}}$ and a component F' of Z_+ such that the sign vector of F' is obtained from that of F by flipping the γ-entry of the latter from -1 to 1.*

Proof. As F is not maximal, there exists a component F'' of Z_+ for which $X_M(F'')$ properly contains $X_M(F)$. Let $\lambda \in F, \mu \in F''$ be points such that the line segment $\nu(t) := (1-t)\lambda + t\mu,\ t \in [0,1]$ does not intersect an element of $\mathcal{L}$ of codimension > 1 in P_+ (such points λ,μ exist). Let t_0 be the smallest t for which $\nu(t)$ does not lie in F, and let V be the unique element of $\mathcal{L}$ containing $\nu(t_0)$. Let $\gamma \in X_M$ be such that $\gamma^\perp \cap P_+ = V$; the ray $\mathbb{R}_+\gamma$ then lies in X_M^{rel}. If γ would be in $X_M(F)$, i.e., if it would be positive on F, then it would be negative on $\nu(t)$ for $t > t_0$, hence in particular γ would not lie in $X_M(F'')$—contrary to our assumption that the latter set contains $X_M(F)$. Hence $\gamma \notin X_M(F)$. The same argument shows that no negative multiple of γ can lie in $X_M(F)$, so that $-\mathbb{R}_+\gamma \notin X_M^{\mathrm{rel}}$. Now let F' be the component of Z entered by $\nu(t)$ at $t = t_0$. Then the above shows that

$$X_M(F') = X_M(F) \cup (\mathbb{R}\gamma \cap X_M),$$

i.e., the sign vectors of F and F' are related as claimed. □

The algorithm above was used to compute the number of connected components of $\mathcal{N}(M^p)$ for $G = \mathrm{SL}_n \times \mathrm{SL}_n$ acting on $M = M_n$ for small values of n; this lead to the description of this null-cone in the Introduction [4].

We conclude this section with some remarks on the combinatorial-computational aspects of our insights so far. Of course, the algorithm above is rather naive. More advanced algorithms for cell enumeration in a real hyperplane arrangement are well known—see, e.g., [1] and the textbook [5]—and could be adapted to our needs. However, I did not find literature addressing the problem of counting only *maximal* cells. More specifically, it makes sense to define a subcomplex of $\mathcal{F}$ as follows: take those cells whose sign vector is *maximal* relative to the entrywise partial order on $\{-1,0,1\}^{X_M^{\mathrm{rel}}} \times \{0,1\}^{\Delta}$ induced by the partial order on $\{-1,0,1\}$ in which $-1 < 1$ is the only strict inclusion. In other words, only cells lying on precisely the same hyperplanes $\gamma^\perp$, $\gamma \in X_M^{\mathrm{rel}} \cup \Delta$ can be compared, and one is smaller than the other if the set of $\gamma \in X_M^{\mathrm{rel}}$ positive on the first cell is contained in the corresponding set for the second cell. These 'maximal' cells form a complex, and the cells of maximal dimension are the ones that we want to count. It would be interesting to see if there is a Zaslavsky-type formula for the number of cells in this complex.

4. Null-cones with few irreducible components

The adjoint representation of G on its Lie algebra $\mathfrak{g}$ has the property that $\mathcal{N}(\mathfrak{g}^p)$ is irreducible for all p; in other words: $c(\mathfrak{g}^p) = 1$ for all p. Using the results of the previous section this is clear: the non-zero weights of T in $\mathfrak{g}$ are the roots, and each of these is either a positive or a negative linear combination of the simple roots. Lemma 3.2 shows that none of them is relevant, so that $Z_+ = P_+^\circ$ has only one connected component.

We now want to find out how many representations share this property with the adjoint representation: with G and M as before, when is $c(M^p) = 1$ for all p? More generally, it is natural to ask, for how many representations M the numbers $c(M^p)$ are uniformly bounded by a prescribed number N. Note that for reductive, non-semisimple G one can construct many such M by taking an arbitrary module and tensoring with an appropriate character of G, to the effect that $\mathcal{N}(M^p) = M^p$ for all p. Hence it makes sense to study this question for semisimple G first. Now for G semisimple of type A_1 one readily finds that $Z_+ = P_+^\circ$ for *all* modules M, so we have to exclude A_1, as well, from our discussion. Using the theory of the previous sections, we can prove the following fundamental result.

Theorem 4.1. *Suppose that* $\operatorname{char} K = 0$*. Let* N *be a natural number and suppose that* G *is a semisimple algebraic group without simple components of type* A_1*. Then there exists a finite subset* S *of* $X(T)$ *such that every rational* G*-module* M *having* $c(M^p) \leq N$ *for all* p *has* $X_M \subseteq S$*.*

The condition $\operatorname{char} K = 0$ will be used for a convenient description of the weight system of irreducible modules. It would be interesting to investigate whether this condition is really necessary; but here we assume from now on that $\operatorname{char} K = 0$.

It is not hard to see that the simply connected cover of G has the same null-cone on M as G, so that we may as well assume that G is simply connected. Also, the number $\lim_{p\to\infty} c(M^p)$ is an ascending function of X_M by the results of the previous sections, so that it suffices for the theorem to prove that only finitely many *irreducible* modules M have $\lim_{p\to\infty} c(M^p) \leq N$. The following lemma further reduces the proof of the theorem to the case of simple G.

Lemma 4.2. *Suppose that* $G = G^1 \times G^2$ *where* G^1, G^2 *are semisimple, and let* M *be an irreducible rational* G*-module. Then* $\lim_{p\to\infty} c_G(M^p) \geq \lim_{p\to\infty} c_{G^1}(M^p)$, *where* $c_{G^1}(M^p)$ *is the number of components of the null-cone in* M^p*, regarded as a* G^1*-module.*

Proof. We write $1 = T^1 \times T^2$, where T^i is a maximal torus in G^1. Accordingly, we have $X = X^1 \times X^2$ for the character groups, $Y = Y^1 \times Y^2$ for the groups of 1-PSGs, and $P_+ = P_+^1 \times P_+^2$. Because M is irreducible, we have $X_M = X_{M_1} \times X_{M_2}$ for some irreducible G^i-modules M_i. As a G_1-module, M is isomorphic to a direct sum of $\dim M_2$ copies of M_1, so that $c_{G^1}(M^p) = c_{G^1}(M_1^{p \dim M_2})$; hence we have to prove that that $\lim_{p\to\infty} c_G(M^p) \geq \lim_{p\to\infty} c_{G^1}(M_1^p)$.

To this end, fix a weight $\gamma^2 \in X_{M_2}$ that lies in the closed cone generated by the *negative* roots for G^2—such a weight always exists by W-stability of X_{M_2}—so that $\langle \gamma^2, \lambda^2 \rangle \leq 0$ for all $\lambda^2 \in P_+^2$. For all $\gamma^1 \in X_{M_1}$ and all $\lambda = \lambda^1 + \lambda^2 \in P_+$ we then have

$$\langle \gamma^1 + \gamma^2, \lambda \rangle > 0 \Rightarrow \langle \gamma^1, \lambda^1 \rangle > 0.$$

Now let μ, μ' be representantives of distinct maximal components of $Z_+^1 = (P_+^1)^\circ \setminus \bigcup_{\gamma \in X_{M_1}} \gamma^\perp$. Then μ and μ' define (not necessarily maximal) halfs $X_M(\mu)$ and $X_M(\mu')$ in X_M. We claim that no half $X_M(\lambda),\ \lambda = \lambda^1 + \lambda^2 \in P_+$, contains both the halfs $X_M(\mu)$ and $X_M(\mu')$. Indeed, suppose that $X_M(\lambda)$ contains $X_M(\mu)$. In particular, we have

$$\langle \gamma^1 + \gamma^2, \mu \rangle > 0 \Rightarrow \langle \gamma^1 + \gamma^2, \lambda^1 + \lambda^2 \rangle > 0$$

for all $\gamma^1 \in X_{M^1}$. But by the choice of γ^2 this implies

$$\langle \gamma^1, \mu \rangle > 0 \Rightarrow \langle \gamma^1, \lambda^1 \rangle > 0,$$

for all $\gamma^1 \in X_{M^1}$, so that $X_{M_1}(\lambda^1)$ contains $X_{M_1}(\mu)$. An identical argument shows that $X_{M_1}(\lambda^1)$ contains $X_{M_1}(\mu')$, but this contradicts the fact that μ and μ' represent distinct maximal components of Z_+^1. We conclude that there are at least as many maximal components in Z_+ as in Z_+^1. □

In what follows G is simply connected and simple and $(.,.)$ is a W-invariant inner product on $X_\mathbb{R}$, which provides a norm $||.||$ on $X_\mathbb{R}$. For $\gamma_0 \in X$ a dominant weight, we write $M(\gamma_0)$ for the irreducible G-module of highest weight γ_0. Let $R \subseteq X$ be the root lattice, i.e., the lattice generated by the α_i, so that $X_{M(\gamma_0)} \subseteq \gamma_0 + R$. We need to find big balls in weight systems of large representations.

Lemma 4.3. *Suppose that* G *is simple and simply connected, and let* A *be a bounded subset of* $X_\mathbb{R}$*. Then there are only finitely many dominant weights* γ_0 *for which* $X_{M(\gamma_0)}$ *does not contain* $(\gamma_0 + R) \cap A$*.*

Proof. It suffices to prove this fact for $A = B_r = \{\gamma \in X_\mathbb{R} \mid ||\gamma|| \leq r\}$ for $r \in \mathbb{R}_+$. Suppose therefore that $X_{M(\gamma_0)}$ does not contain $(\gamma_0 + R) \cap B_r$; we will show that this excludes all but finitely many highest weights. It is well known—and here we use $\operatorname{char} K = 0$—that $X_{M(\gamma_0)}$ is the intersection with $\gamma_0 + R$ of the convex hull of the W-orbit of γ_0 [6]; denote this convex hull by C. By assumption, C does not contain the ball B_r, and if $\alpha \in B_r$ is a point not belonging to C, then by convexity of C there exists an affine hyperplane H through α not intersecting C. Let β be the shortest vector of H, so that $||\beta|| \leq ||\alpha|| \leq r$ and H is the set of vectors whose inner product with β equals $||\beta||^2$. As C does contain 0—the sum of the W-orbit of γ_0 is 0—we find that C lies on the side of H where the inner product with β is small; in particular, we have $(\gamma_0, \beta) < ||\beta||^2 \leq r^2$.

By the W-action we may in addition assume that β is in the positive Weyl chamber, i.e., has non-negative inner product with all simple roots (note that this is a different Weyl chamber from P_+, which lives in the dual space $Y_{\mathbb{R}}$). Now the positive Weyl chamber is sharp-angled in the sense that any two non-zero elements in it have positive inner product—this can be checked by computing the inner products of the fundamental weights as listed, for instance, in [3]; we use here that the group is simple!—and it follows from this that there is a positive constant a with the property that $(\delta_1, \delta_2) \geq a||\delta_1||||\delta_2||$ for all δ_1, δ_2 in the positive Weyl chamber. But then also $(\gamma_0, \beta) \geq a||\gamma_0||||\beta||$ and combining this with the inequality obtained above we find

$$||\gamma_0|| < (1/a)||\beta|| \leq r/a,$$

so that, indeed, only finitely many highest weights have this property. □

The key tool in our proof of Theorem 4.1 is the following lemma.

Lemma 4.4. *Let G be a connected, reductive algebraic group and let M be a rational G-module. Suppose that X_M contains strings of relevant weights of the form*

$$A = \{\gamma, \gamma + \alpha, \dots, \gamma + (N-1)\alpha\} \text{ and}$$
$$B = \{-\gamma - c\alpha, -\gamma - (c+1)\alpha, \dots, -\gamma - (c+N-1)\alpha\}$$

where α is a non-negative (integer) linear combination of the simple roots and where $c \in [0, 1)$. Then $\lim_{p\to\infty} c(M^p) > N$.

Proof. The hyperplanes $\beta^\perp$ with $\beta \in A$ cut the open Weyl chamber P_+° into $N+1$ connected components $C_0, \dots, C_N$, where $\lambda \in C_j$ is equivalent to

$$\langle \gamma + k\alpha, \lambda \rangle < 0 \text{ for all } k = 0, \dots, j-1 \text{ and}$$
$$\langle \gamma + k\alpha, \lambda \rangle > 0 \text{ for all } k = j, \dots, N-1.$$

Here we used $\langle \alpha, \lambda \rangle > 0$ for all $\lambda \in P_+$ and the fact that the weights in A are relevant.

Now if $c = 0$, then $B = -A$ and the hyperplanes annihilating the weights in B do not refine the subdivision of P_+°. If, on the other hand, $c > 0$, then the inequalities

$$\langle \gamma + (j+1)\alpha, \lambda \rangle > -\langle -\gamma - (c+j)\alpha, \lambda \rangle > \langle \gamma + j\alpha, \lambda \rangle,$$

for $\lambda \in P_+^\circ$ and all j, show that for $j = 0, \dots, N-1$ the hyperplane $(-\gamma - (c+j))^\perp$ cuts C_{j+1} into two components, while not meeting the other C_k. In total, the hyperplanes $\beta^\perp$ with $\beta \in A \cup B$ therefore cut P_+° into $2N+1$ connected components. Here we used that the weights in B are relevant.

Regardless of whether $c = 0$ or $c \neq 0$, the hyperplanes $\beta^\perp$ with $\beta \in A \cup B$ cut P_+° into connected components of which $N+1$ are maximal relative to $A \cup B$—namely those where λ satisfies

$$X_M(\lambda) \cap (A \cup B) = \{-\gamma - c\alpha, \dots, -\gamma - (c+j-1)\alpha, \gamma + j\alpha, \dots, \gamma + (N-1)\alpha\}$$

for some $j \in \{0, \dots, N\}$. By taking into account the other weights of M, as well, the number of maximal components can only increase. The lemma now follows from the results of the previous sections. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. As noted before, we may assume that G is simply connected and we may concentrate on irreducible representations. By Lemma 4.2 we may in addition assume that G is simple. We now show that in irreducible representations whose highest weight is sufficiently large the existence of α-strings as in Lemma 4.4 among the relevant weights is unavoidable. Indeed, set

$$D := \sum_i [0,1)\alpha_i;$$

as G is semisimple, D is a fundamental domain for R in $X_\mathbb{R}$. Let $d \in \mathbb{N}$ be the (finite) index of R in X. Now choose a relevant virtual weight $\gamma' \in X_\mathbb{R}$ for which the closure $\overline{\gamma' + 2D + (N-1)dD}$ lies entirely in the set of relevant virtual weights. This is possible, since the relevant virtual weights form a non-empty open subset in $X_\mathbb{R}$ that is stable under multiplication with $\mathbb{R} \setminus \{0\}$ (see Lemma 3.2)—here we use that G is not of type A_1, in which case there are no relevant weights.

For all but finitely many dominant weights $\gamma_0 \in X$ the weight system of $M(\gamma_0)$ contains the set

$$(\overline{\gamma' + 2D + (N-1)dD} \cup \overline{-\gamma' - 2D - (N-1)dD}) \cap (\gamma_0 + R)$$

by Lemma 4.3. Suppose that this is the case. As D is a fundamental domain for R, $\gamma' + D$ contains a unique element γ of $\gamma_0 + R$ and also $-\gamma - D$ contains a unique element δ of $\gamma_0 + R$; hence $\delta \in -\gamma' - 2D$. Now if $\delta = -\gamma$—in particular, this will be the case if $d = 1$, as then R equals X—then we let α be an arbitrary simple root and we set $c := 0$. Otherwise, we set $\alpha := d(-\gamma - \delta) \in dD$ and $c := 1/d < 1$. In either case, we find $j\alpha \in \overline{jdD}$ for all j. By the assumption on γ_0, the weight system $X_{M(\gamma_0)}$ contains the α-strings

$$\gamma, \gamma + \alpha, \ldots, \gamma + (N-1)\alpha$$

and

$$\delta = -\gamma - c\alpha, -\gamma - c\alpha, \ldots, -\gamma - (c + N - 1)\alpha,$$

that is, two α-strings as needed for Lemma 4.4. □

Regarding the first question of this section: it would be interesting to classify, for simple G, all G-representations M for which $\mathcal{N}(M^p)$ is irreducible for all $p > 0$. Surprisingly, the easy sufficient condition for *reducibility* obtained by taking $N = 1$ in Lemma 4.4—namely, the existence of relevant weights γ and δ in $X(M)$ with the property that $-\gamma - \delta$ is a positive linear combination of the simple roots—seems strong enough to classify such M for simple G. I conclude this paper with a conjecture, partially based on computations in `LiE` [9].

Conjecture 4.5. *Suppose that G is a simple, simply connected algebraic group in characteristic 0 and of rank ≥ 2; and let M be a finite-dimensional, rational G-module for which $\mathcal{N}(M^p)$ is irreducible for all p. Let L be the set of highest weights of T in M, but leave out the adjoint weight and the trivial weight. Then (G, L) is in the following list (relative to the ordering of simple roots and fundamental weights as in* [3]*).*

(1) *$G = A_2$ and $L \subseteq \{\omega_1, 3\omega_1\}$ (standard representation V and $S^3(V)$) or $L \subseteq \{\omega_2, 3\omega_2\}$ (V^* and $S^3(V^*)$).*
(2) *$G = A_r$ with $r > 2$ and L is a subset of $\{\omega_1\}$ (standard) or of $\{\omega_r\}$ (dual standard).*

(3) *$G = B_2$ and L is a subset of $\{\omega_1, \omega_2, 2\omega_1\}$ (standard, 4-dimensional, or quadratic forms modulo the defining form).*
(4) *$G = B_r$ with $r \geq 3$ and L is a subset of $\{\omega_1, 2\omega_1\}$ (standard, or quadratic forms modulo the defining form).*
(5) *$G = C_r$ with $r \geq 3$ and L is a subset of $\{\omega_1, \omega_2\}$ (standard, or skew symmetric bilinear forms modulo the defining form).*
(6) *$G = G_2$ and L is a subset of $\{\omega_1, 2\omega_1\}$.*
(7) *$G = F_4$ and L is a subset of $\{\omega_4\}$ (the 26-dimensional module).*
(8) *$G = E_6, E_7,$ or E_8 and $L = \emptyset$.*

Conversely, for all pairs (G, M) for which (G, L) is in the list above, $\mathcal{N}(M^p)$ is irreducible for all p.

The last statement of this conjecture is readily verified case by case. Moreover, for G not of type B_n, C_n, E_7, E_8, F_4, or G_2 the conjecture is true, and verified as follows: For these G any module M is self-dual, so that $X_M = -X_M$. By the results of the previous sections $c(M^p)$ is then the number of connected components of Z_+. Now if X_M contains a weight γ that is *not* a multiple of a root, then $\gamma^\perp$ meets some open Weyl chamber in $Y_\mathbb{R}$, and hence for some W-conjugate $\delta \in X_M$ of γ the hyperplane $\delta^\perp$ cuts P_+° into two parts. But then Z_+ has at least 2 connected components. Therefore, any M with $\mathcal{N}(M^p)$ irreducible for all p must have $X_M \subseteq \bigcup_{\alpha \in \Delta} \mathbb{R}\alpha$—and, conversely, for any M with this property $\mathcal{N}(M^p)$ is irreducible for all p. It is not hard to determine, for each of the types above, the dominant weights γ_0 for which $X_{M(\gamma_0)}$ consists entirely of root multiples, and this results in the list above.

In the remaining two cases, A_n and E_6, however, the existence of relevant weights does *not* imply the reducibility of $\mathcal{N}(M^p)$, and more subtle reasoning would be needed to prove the conjecture: For dominant weights γ_0 not appearing in the list above, one would have to exhibit relevant weights γ and δ in $X_{M(\gamma_0)}$ such that $-\gamma - \delta$ is a positive linear combination of the simple roots. Though elementary, this task seems combinatorially rather intricate, and the conjecture above is merely supported by computations exhibiting such γ and δ in small modules M for E_6 and for small A_n.

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